Self-similar fast reaction limit of reaction diffusion systems with nonlinear diffusion

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Abstract

In this thesis, we present an approach to characterising fast-reaction limits of systems with nonlinear diffusion, when there are either two reaction-diffusion equations, or one reaction-diffusion equation and one ordinary differential equation on unbounded domains. Here, we replace the terms of the form u_{xx} in usual reaction-diffusion equation, which represent linear diffusion, by terms of form $\phi(u)_{xx}$, representing nonlinear diffusion. For appropriate initial data, in the fast-reaction limit $k \to \infty$, spatial segregation results in the two components of the original systems each converge to the positive and negative points of a self-similar limit profile $f(\eta)$, where $\eta = \frac{x}{\sqrt{t}}$, that satisfies one of four ordinary differential systems. The existence of these self-similar solutions of the $k \to \infty$ limit problems is proved by using shooting methods which focus on a, the position of the free boundary which separates the regions where the solution is positive and where it is negative, and γ, the derivative of $-\phi(f)$ at $\eta = a$. The position of the free boundary gives us intuition how one substance penetrates into the other, so for specific forms of nonlinear diffusion, the relationship between the given form of the nonlinear diffusion and the position of the free boundary is also studied.

Key Words: Nonlinear diffusion; Reaction diffusion problem; Fast reaction; Free boundary; Self-similar solution.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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The University's ethical procedures have been followed and, where appropriate, that ethical approval has been granted.

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Notation

$\mathbb{R}^+ := \{x : 0 \le x < \infty\}$

$s^+ := \max\{0, s\}$

$s^- := \min\{0, s\}$

$S_T := \{(x,t) : 0 < x < \infty,\ 0 < t < T\}$

$Q_T := \{(x,t) : x \in \mathbb{R},\ 0 < t < T\}$

$\Omega_\kappa := \{\xi \in W^{1,2}((0,\kappa))|\ \xi = 0 \text{ at } x = 0\}$

$\mathcal{F}_T^R := \{\xi \in C^1\left([0,R] \times [0,T]\right) \mid \xi(0,t) = \xi(\cdot,T) = 0 \text{ for } t \in (0,T)\}$

$\mathcal{F}_T := \{\xi \in C^1(S_T) :\ \xi(0,t) = \xi(\cdot,T) = 0 \text{ for } t \in (0,T) \text{ and } \operatorname{supp}\xi \subset [0,J] \times [0,T]$
for some $J > 0\}$

$\hat{\mathcal{F}}_T := \{\xi \in C^1(Q_T) :\ \xi(\cdot,T) = 0 \text{ and } \operatorname{supp}\xi \subset [-J,J] \times [0,T]$
for some $J > 0\}$

a: the position of the free boundary which separates the regions where $f(\eta) > 0$ and where $f(\eta) < 0$ and $\lim_{\eta \nearrow a} f(\eta) = 0 = \lim_{\eta \searrow a} f(\eta)$.

$\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta)$.

$f(\eta; a, \gamma)$: self-similar solution of the $k \to \infty$ limit problems for given a, γ.

$$b(a,\gamma) := \begin{cases} \lim_{\eta \to 0} f(\eta; a, \gamma), & \text{half-line case,} \\ \lim_{\eta \to -\infty} f(\eta; a, \gamma), & \text{whole-line case.} \end{cases}$$

$d(a,\gamma) := \lim_{\eta \to \infty} f(\eta; a, \gamma)$.

Chapter 1

Introduction

The reaction diffusion problem

$$\begin{cases} u_t = u_{xx} - kuv, & (x,t) \in (0,\infty) \times (0,T) \\ v_t = -kuv, & (x,t) \in (0,\infty) \times (0,T) \\ u(0,t) = U_0, & \text{for } t \in (0,T) \\ u(x,0) = 0,\ v(x,0) = V_0, & \text{for } x \in (0,\infty) \end{cases} \tag{1.1}$$

originates from the chemical reaction

$$A + B \xrightarrow{k} C,$$

occurring in a semi-infinite region. Here u represents concentration of the chemical A which may disperse in the substrate through diffusion, v represents concentration of the immobile substrate B, and k is the rate constant of the reaction (which is positive). The chemical reaction can be modelled for simplicity by the one-dimensional spatial domain $(0,\infty)$ with $u = U_0$ at the surface $x = 0$. That u and v are nonnegative is natural since they typically correspond to concentration of chemical substances.

In [17], Hilhorst, van der Hout and Peletier studied the asymptotic behaviour of k-dependent solutions (u^k, v^k) of (1.1) as $k \to \infty$ (i.e. the reaction is very fast). They established a free boundary problem which is satisfied in the limit when solution (u^k, v^k) converges to a self-similar limit $(u, v)\left(\frac{x}{\sqrt{t}}\right)$ as $k \to \infty$. The free boundary has the form $x = a\sqrt{t}$, where $a > 0$ and divides the area in which the mobile chemical A is present from the area where A is absent. The fast-reaction limit of (1.1) can be motivated by the study of penetration of radio-labeled antibodies into tumourous tissue since the attachment of antibodies to antigens in the tissue may react very fast.

Modelling can give rise to other systems related to (1.1) such that the fast-reaction limit in which one mobile substance invades a *mobile* substrate. Among other problems, Crooks and Hilhorst [10] studied the system analogous to (1.1) when reactant u and substrate v are both mobile, for example, when carbonic acid penetrates into water. In this case, the substrate will diffuse, which is modelled by introducing a term $d_v v_{xx}$ where $d_v > 0$. The paper [10] is concerned with the free boundary problems in the limit that $k \to \infty$ in four cases: $d_v > 0$ with two mobile reactants, $d_v = 0$ with one mobile and one immobile reactant, problems defined on the spatial domain $(0, \infty)$ as in (1.4) and also on the whole real line $\mathbb{R}$, which can arise, for instance, in modelling neutralisation of an acid and a base that are initially separated. In all four cases, the free boundary has the form $x = a\sqrt{t}$ where the constant a is determined by a different equation in each case and plays an important role in characterising the rate of penetration of one substance into the other in the limit $k \to \infty$. When the problem is considered on the spatial domain $\mathbb{R}$ with $d_v > 0$, the constant a in the corresponding limit is not necessarily positive. Note that when $a > 0$, substance u penetrates into substance v, while on the other hand, v penetrates into u when $a < 0$. For each of the problems with $d_v \geq 0$ on both the spatial domains $\mathbb{R}$ and $(0, \infty)$, an explicit formula is given in [10] for the self-similar limit function.

Nonlinear diffusion is needed in certain modelling scenarios to describe processes involving fluid flow, heat transfer or diffusion. For instance, it can describe the flow of an isentropic gas through a porous medium [8]. The analogue of (1.1) with nonlinear diffusion in bounded multi-dimensional domains is studied in [18] by Hilhorst, van der Hout and Peletier. They consider the substrate u with nonlinear diffusion modelled with a term $\Delta\phi(u)$, where $\phi(u) = \int_0^u D(s)\mathrm{d}s$ and D is the diffusivity of the medium. Under assumptions in [18], $D(s)$ may vanish at $s = 0$, so the equation for u need not be uniformly parabolic. Thus [18] focuses on weak solutions since it is possible that the system studied has no classical solution. In studying of the multi-dimensional limiting free boundary problems in [18], the free boundary $\Gamma(t)$ of the limit problem is assumed as a smooth surface that lies entirely within the bounded domain and varies smoothly with t.

Unlike in the case of linear diffusion, explicit self-similar solutions to the limit problems obtained in the nonlinear diffusion case are not readily available. We briefly mention here two distinct alternative approaches that have been used previously to investigate self-similar solutions in the one-dimensional nonlinear diffusion case. Atkinson and Peletier [2] studied a self-similar solution on a bounded domain by looking first at the initial value problem starting at the free boundary, and then investigating how the value of this solution at $\eta = 0$ depends on the position of the free boundary. Similar problems but on unbounded domain are studied by Craven and Peletier [7] using a shooting method. These ideas will form the starting point for proving existence of self-similar solutions for the limit problems derived in this thesis in Chapter 3.

A prototype for the form of nonlinear diffusion considered with this thesis is $(u^m)_{xx}$, where $m > 1$. Throughout this thesis, we will consider nonlinear diffusion terms of the form $\phi(u)_{xx}$, where the function $\phi \in C^2(\mathbb{R})$, ϕ and ϕ' are assumed to be strictly increasing with

$$\phi(s) > 0 \text{ as } s > 0 \text{ and } \phi'(s) = \phi(s) = 0 \text{ when } s = 0. \tag{1.2}$$

In studying existence of self-similar solutions in Chapter 3 and Section 4.4, we will also require that ϕ satisfies

$$\int_0^1 \frac{\phi'(f)}{f}\mathrm{d}f < \infty \quad \text{and} \quad \int_1^\infty \frac{\phi'(f)}{f}\mathrm{d}f = \infty. \tag{1.3}$$

Here the self-similar solutions of the limit problems with nonlinear diffusion have the same ansatz as in the linear diffusion case, in the sense that $w(x,t) = f(\eta)$ where $\eta = x/\sqrt{t}$, see [2, 7]. Note that this contrasts with the famous family of special solutions of the porous medium equation $u_t = (u^m)_{xx}$ known as Barenblatt solutions, that represent heat release from a point source and take as initial data a Dirac mass, where m appears explicitly in the solution ansatz. In this thesis, our initial data is bounded and we always consider self-similar solutions of the form $f(x/\sqrt{t})$, and will study how the profile f of these solutions is affected by the nonlinear diffusion.

We treat two pairs of problem with nonlinear diffusion terms on the spatial domains $\mathbb{R}^+$ and $\mathbb{R}$. The first pair of problems defined on the half-strip $S_T := \{(x,t) : 0 < x < \infty, 0 < t < T\}$, one with $\varepsilon > 0$ and the other with $\varepsilon = 0$, are

$$\begin{cases} u_t = \phi(u)_{xx} - kuv, & (x,t) \in (0,\infty) \times (0,T), \\ v_t = \varepsilon\phi(v)_{xx} - kuv, & (x,t) \in (0,\infty) \times (0,T), \\ u(0,t) = U_0, \quad \varepsilon\phi(v)_x(0,t) = 0, & \text{for} \quad t \in (0,T), \\ u(x,0) = u_0^k(x), \quad v(x,0) = v_0^k(x), & \text{for} \quad x \in \mathbb{R}^+. \end{cases} \tag{1.4}$$

As in [17] kuv is the contribution of a chemical reaction where k determines the reaction rate. We define, as in [10], the initial data for the limiting self-similar solutions as

$$u_0^\infty = \begin{cases} U_0 & x = 0, \\ 0 & x > 0, \end{cases} \qquad v_0^\infty = \begin{cases} 0 & x = 0, \\ V_0 & x > 0, \end{cases}$$

which equal constant initial conditions on the half-line in [17], where U_0 and V_0 are positive constants, and choose the initial data u_0^k, v_0^k that satisfy

(i) $u_0^k, v_0^k \in C^2(\mathbb{R}^+)$;

(ii) $0 \le u_0^k \le U_0,\ 0 \le v_0^k \le V_0$;

(iii) $u_0^k \to u_0^\infty,\ v_0^k \to v_0^\infty$ in $L^1(\mathbb{R}^+)$ as $k \to \infty$.

(iv) For each $r > 0$, there exists a continuous function $\omega_r : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\omega_r(\mu) \to 0$ as $\mu \to 0$ and

$$\|u_0^k(\cdot + \delta) - u_0^k(\cdot)\|_{L^1((r,\infty))} + \|v_0^k(\cdot + \delta) - v_0^k(\cdot)\|_{L^1((r,\infty))} \le \omega_r(\delta),$$

for all $k > 0$, $\delta < \frac{r}{4}$.

For both $\varepsilon = 0$ and $\varepsilon > 0$, we will prove the existence and uniqueness of weak solutions (u^k, v^k) of problem (1.4) for every $k > 0$, and study the asymptotic behaviour of (u^k, v^k) as $k \to \infty$. As we will see, the limits u of u^k and v of v^k are separated by a free boundary and given by the positive and negative parts respectively of a function w, that is

$$u = w^+ \text{ and } v = -w^-,$$

where $s^+ = \max\{0, s\}$ and $s^- = \min\{0, s\}$. We show that this limit function w has one of two self-similar forms, depending on whether $\varepsilon > 0$ or $\varepsilon = 0$. The function $f : \mathbb{R}^+ \to \mathbb{R}$ decribes a self-similar limit solution such that $w(x,t) = f(\eta)$ where $\eta = x/\sqrt{t}$ for $(x,t) \in S_T$. There is a free boundary at $\eta = a$ with $f(\eta) > 0$ when $\eta < a$ and $f(\eta) < 0$ when $\eta > a$ and the self-similar solution $f(\eta)$ satisfies the boundary conditions $f(a) = 0$ and

$$\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta). \tag{1.5}$$

When $\varepsilon > 0$, the existence of self-similar solutions is proved by using a two-parameter shooting methods focusing on a and γ. When $\varepsilon = 0$, γ has a specific form, namely

$$\gamma = \frac{aV_0}{2},$$

and the existence of self-similar solutions is proved by a one-parameter shooting, since γ depends on a.

The second pair of problems is defined on the strip $Q_T := \{(x,t) : x \in \mathbb{R},$ $0 < t < T\}$, one with $\varepsilon > 0$ and the other one with $\varepsilon = 0$ are

$$\begin{cases} u_t = \phi(u)_{xx} - kuv, & (x,t) \in \mathbb{R} \times (0,T), \\ v_t = \varepsilon\phi(v)_{xx} - kuv, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0^k(x), \quad v(x,0) = v_0^k(x), & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where we define, as in [10] that

$$u_0^\infty = \begin{cases} U_0 & x < 0, \\ 0 & x > 0, \end{cases} \qquad v_0^\infty = \begin{cases} 0 & x < 0, \\ V_0 & x > 0, \end{cases}$$

with U_0, V_0 positive constants, k as in (1.4) and initial data u_0^k, v_0^k satisfy

(i) $u_0^k, v_0^k \in C^2(\mathbb{R})$;

(ii) $0 \leq u_0^k \leq U_0,\ 0 \leq v_0^k \leq V_0$;

(iii) $u_0^k \to u_0^\infty,\ v_0^k \to v_0^\infty$ in $L^1(\mathbb{R})$ as $k \to \infty$.

(iv) There exists a continuous function $\omega : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\omega(\mu) \to 0$ as $\mu \to 0$ and

$$\|u_0^k(\cdot + \delta) - u_0^k(\cdot)\|_{L^1(\mathbb{R})} + \|v_0^k(\cdot + \delta) - v_0^k(\cdot)\|_{L^1(\mathbb{R})} \leq \omega(\delta),$$

for all $k > 0$, $\delta \in \mathbb{R}$.

Note that for simplicity, we use the same notation u_0^∞, v_0^∞ for both half-line and whole line functions.

We again consider both the case of two mobile reactants where $\varepsilon > 0$, and the case of one mobile and one immobile reactant, when $\varepsilon = 0$. Similar to the half-line case, we also prove the existence and uniqueness of weak solutions (u^k, v^k) of problem (1.6), and study the convergence to self-similar limit profiles (u, v) as $k \to \infty$. We use arguments similar to those in half-line case to prove the existence of a self-similar limit solution f of (1.6). If $\varepsilon > 0$, we may have $a < 0$, $a = 0$ and $a > 0$ where $f(a) = 0$, since a is not necessarily positive in the whole-line case.

The work of this thesis continues and extends earlier studies of fast-reaction limits [10][17][18] and self-similar solutions with nonlinear diffusion [2][7], by introducing the nonlinear function ϕ, in both the case of two mobile reactants ($\varepsilon > 0$) in addition to that of one mobile reactant ($\varepsilon = 0$) and in considering the whole-line problem (1.4) in addition to the half-line problem (1.6). In [18], the existence of weak solutions is proved by looking at a sequence of uniformly parabolic problems in which $\phi_n'(u) \geq \frac{1}{n}$ and studying the solutions in the limit as $n \to \infty$. We exploit some ideas and an iterative method from [18], but our domains are unbounded and when $\varepsilon > 0$, the equations for both u and v of (1.4) and (1.6) have nonlinear diffusion and are not uniformly parabolic. In the problems treated in [10], where the diffusion is linear and the problems are studied in unbounded domains, a series of cut-off functions and auxiliary functions are introduced to prove various estimates of u^k, v^k that are useful in studying the $k \to \infty$ limit. Here, we consider $\phi(u^k), \phi(v^k)$ rather than u^k, v^k and in order to deal with the nonlinear diffusion, alternative methods and additional procedures are needed. Note that we know of no explicit self-similar solutions for the $k \to \infty$ limit problems with nonlinear diffusion that are obtained here. In previous studies [2, 7], they treated the single equation where the solutions were always non-negative. In this thesis, we have sign-changing solutions since the free boundary separates regions where the solutions are positive and where the solutions are negative. Here, our self-similar solutions satisfy a certain equation when they are positive, and a different equation where they are negative. We exploit ideas from [2, 7] and investigate our self-similar limit problems that involve these two equations.

The rest of the thesis is organised as follows. In Chapter 2, we study the half-line problem (1.4), starting with the uniqueness of weak solutions for (1.4). Under the assumptions on ϕ in (1.2), the equations for u, v need not be uniformly parabolic when $\varepsilon > 0$, so the existence of weak solution for (1.4) are proved in Theorem 2.9 by an iterative method. Section 2.3 is concerned with passing to the limit as $k \to \infty$ of the weak solutions (u^k, v^k), via some *a priori* estimates and a key bound on ku^kv^k in $L^1(S_T)$, independent of k and $\varepsilon \geq 0$ which is proved in Theorem 2.31. The $k \to \infty$ limit (u^k, v^k) of (1.4) is characterised as a self-similar solution of the problem first in Theorem 3.3 when $\varepsilon > 0$, and then in Theorem 3.11 when $\varepsilon = 0$, the existence of self-similar solutions is proved later in Section 3.3 and 3.4. Chapter 3 focuses on properties of the parameters a in the studying of the self-similar solution f, the position of free boundary which separates the region where the solution is positive and where it is negative, and $-\gamma$, the negative gradient of $\phi(f)$

at $\eta = a$, and prove some preliminary results that are useful in deducing existence of self-similar solutions. We split the self-similar solutions into two parts depending on whether $\eta < a$ or $\eta > a$, and study the properties of $b(a, \gamma), d(a, \gamma)$, defined respectively as the value of f when $\eta = 0$ and as the limit of f when $\eta \to \infty$, of the self-similar solution $f(\eta)$. Then the existence of self-similar solutions when $\varepsilon > 0$ is proved in Section 3.3 by using a two-parameter shooting method, shooting from $\eta = a$ with γ, the derivatives of $-\phi(f)$ at a, to U_0 and $-V_0$. When $\varepsilon = 0$, the existence of self-similar solutions is proved in Section 3.4 by using a one-parameter shooting method. Chapter 4 contains the whole-line counterparts of the study of the half-line problem in Chapters 2-3. In Chapter 5, we consider a specific family of $\phi'(f) = f^{m-1}$ where $m > 2$ is a constant, and investigate how the free boundary position a is affected by m. Note that with fixed U_0, V_0, there exists a unique self-similar solution which determines a and γ. At the end of Chapter 5, we prove some further results under the additional conditions that $U_0, V_0 < 1$ and $m \geq 2$. In particular, if $\varepsilon = 0$, we find that if $m_1 > m_2$, then $a_{m_1} < a_{m_2}$ which is proved in Theorem 5.6. This result indicates that when m getting smaller, one substance penetrates into the other faster. If $\varepsilon > 0$, the relationship between a and m depends on different cases that $\gamma_{m_1} > \gamma_{m_2}$ or $\gamma_{m_1} < \gamma_{m_2}$ when $a_{m_1} < a_{m_2}$. Note that it is not clear whether or not the results in Section 5.1.1 can be extended to $U_0, V_0 \geq 1$.

There has been a lot of activity on related research problems in recent years, some of which has been useful for this research. We briefly discuss two papers concerned with fast-reaction limits for systems modelling competing species. In [16], Hilhorst, Martin and Mimura considered a competition-diffusion system for two competing species in multi-dimensional bounded spatial domains and studied the behaviour of the free boundary that arises in the fast reaction limit. They provided a strong formulation of the fast reaction limit problem under some regularity assumption on the free boundary and derived conditions that are satisfied on the free boundary. We will exploit the idea to obtain the free boundary conditions of the limit problems in Theorem 3.1. Similar problems but with inhomogeneous Dirichlet boundary conditions are investigated in [9], where Crooks, Dancer, Hilhorst, Mimura and Ninomiya studied the $k \to \infty$ limit problem and gave numerical computations of some two-dimensional patterns of the reaction-diffusion system and of the limiting free boundary problem. The method in [9] for estimating differences in space translates on bounded spatial domains is used in Lemma 2.12.

In studying self-similar solutions, various previous works could potentially be helpful in extending this research further. We mention two papers, [5] and [24], both of which study the self-similar solution $u(x,t) = f(x/\sqrt{t})$ of $u_t = (|u|^{m-1}u_x)_x$ where $m > 0$, by looking at a transformed problem. In [5], Bouillet and Gomes related the self-similar solution satisfying equation $-\frac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]'$, and a singular elliptic equation

$$-yy'' = 2f^{m-1}, \tag{1.7}$$

where $y = \int_0^f \eta(s)\mathrm{d}s$. Duijn, Gomes and Zhang Hongfei studied the solution of transformed problem (1.7), where the flux $\lambda = \frac{1}{2}y(0)$ considered in [24] for single equation plays the similar role as γ from earlier in (1.5) and the numerical results given in [24] suggest λ is decreasing in m, which can be seen in Theorem 5.6 when $\varepsilon = 0$. When $\varepsilon > 0$, it is not clear how to prove this rigorously, but it gives valuable intuition in studying relationship of a, γ and m.

One interesting potential extension of this work is investigating convergence of self-similar solutions when there are two different ϕ for the two components, for example, one with nonlinear diffusion term $\phi(u)_{xx}$ and another one with linear diffusion term v_{xx}. From [10], we can give the self-similar solution when $\eta > a$, namely

$$f(\eta) = -V_0 - Ae^{-\frac{a^2}{4}} \int_{\eta}^{\infty} e^{-\frac{s^2}{4}} \mathrm{d}s,$$

where $A = \lim\limits_{\eta \searrow a} f'(\eta)$ satisfies the free boundary condition

$$\lim_{n \searrow a} f'(\eta) = \lim_{\eta \nearrow a} \phi'(f(\eta)) f'(\eta).$$

Another interesting investigation would be self-similar limit problems with nonlinear diffusion studied in multi-dimensional spatial domains. The self-similar solution $f(\eta)$ in multi-dimensions may take the same form as the one-dimensional problems in this thesis, where $\eta = \frac{x}{\sqrt{t}}$, and $f(\eta) > 0$ satisfies the equations

$$\Delta\phi(f) + \frac{1}{2}\eta\nabla f = 0,$$

see [25, p.406]. Again, it is important to study the set of η on which $f(\eta) = 0$ in multi-dimension, since it separates the region where $f > 0$ from where $f < 0$. Furthermore, in the study of self-similar limit problems in one-dimension, it would be interesting to derive sufficient conditions that ensure $a > 0$ or $a < 0$, since the sign of a determines which substance penetrates into the other. The extension of this to higher dimensions would be to derive sufficient conditions for the set $\{\eta : f(\eta) = 0\}$ to have a particular form.

Chapter 2

The half-line case: Problem (1.4)

In this chapter, we prove the existence and uniqueness of weak solutions of Problem (1.4) when $\varepsilon > 0$ in Section 2.1 by first studying an approximate problem on the bounded spatial domain $(0, R)$ and then studying the limit when $R \to \infty$. In Section 2.2, we prove some *a priori* bounds that will be used to prove the existence and uniqueness of weak solutions when $\varepsilon = 0$ and to study the $k \to \infty$ limit problems both when $\varepsilon > 0$ and $\varepsilon = 0$ in Section 2.3. Our strategy takes advantage of some ideas from [10] and [18].

2.1 Existence and uniqueness of weak solutions for $\varepsilon > 0$

Let $\varepsilon > 0$. We consider first an approximate problem to (1.4). Given $R > 1$, consider the problem

$$\begin{cases} u_t = \phi(u)_{xx} - kuv, & \text{in } (0,R)\times(0,T), \\ v_t = \varepsilon\phi(v)_{xx} - kuv, & \text{in } (0,R)\times(0,T), \\ u(0,t) = U_0, \quad \phi(v)_x(0,t) = 0, & \text{for} \quad t \in (0,T), \\ \phi(u)_x(R,t) = 0, \quad \phi(v)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ u(x,0) = u_{0,R}, \quad v(x,0) = v_{0,R}, & \text{for} \quad x \in (0,R), \end{cases} \tag{2.1}$$

where $u_{0,R}, v_{0,R} \in C^2(\mathbb{R}^+)$ are such that $0 \le u_{0,R} \le U_0$, $0 \le v_{0,R} \le V_0$ and

$$u_{0,R} = u_0^k \beta^R, \qquad v_{0,R} = -(V_0 - v_0^k)\beta^R + V_0, \tag{2.2}$$

where the family of cut-off functions $\beta^R \in C^\infty(\mathbb{R}^+)$ with $R > 1$ are defined as

$$\beta^R = \begin{cases} 1 & x \le R-1, \\ \beta^1(x+2-R) & x \ge R-1. \end{cases}$$

with $\beta^1 \in C^\infty(\mathbb{R}^+)$ is a non-negative cut-off function such that $0 \le \beta^1(x) \le 1$ for all $x \in \mathbb{R}^+$, $\beta^1(x) = 1$ when $x \le 1$ and $\beta^1(x) = 0$ when $x \ge 2$.

Since $\phi'(s)$ may vanish at $s = 0$, the equations for u and v as $\varepsilon > 0$ in problem (2.1) need not be uniformly parabolic and it is possible that there is no classical solution. Thus we are led to introduce a notion of a weak solution.

Now define

$$\Omega_R := \left\{ \alpha \in W^{1,2}(0,R) | \ \alpha = 0 \text{ at } x = 0 \right\}, \tag{2.3}$$

and let $\hat{u} \in C^\infty(\mathbb{R}^+)$ be a smooth function that $\hat{u} = U_0$ when $x = 0$ and $\hat{u} = 0$ when $x > 1$.

Definition 2.1. *A pair* $(u_R, v_R) \in L^\infty((0,R)\times(0,T)) \times L^\infty((0,R)\times(0,T))$ *is called a weak solution of (2.1) if*

(i) $\phi(u_R) \in \phi(\hat{u}) + L^2(0,T;\Omega_R)$, $\phi(v_R) \in L^2(0,T;W^{1,2}(0,R))$;

(ii) (u_R, v_R) *satisfies*

$$\begin{aligned}
&\int_0^R u_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R u_R\xi_t\mathrm{d}x\mathrm{d}t \\
=&\int_0^T\int_0^R \phi(u_R)_x\xi_x\mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R v_R\mathrm{d}x\mathrm{d}t, \\
&\int_0^R v_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R v_R\xi_t\mathrm{d}x\mathrm{d}t \\
=&\int_0^T\int_0^R \varepsilon\phi(v_R)_x\xi_x\mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R v_R\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\xi \in \mathcal{F}_T^R := \{\xi \in C^1([0,R]\times[0,T]) \mid \xi(0,t) = \xi(\cdot,T) = 0 \text{ for } t \in (0,T)\}$.

We use the following comparison theorem for (2.1) to prove the uniqueness of the weak solution of (2.1).

Lemma 2.2. *Suppose that* $\varepsilon \geq 0$ *and* $(\overline{u_R}, \overline{v_R})$, $(\underline{u_R}, \underline{v_R})$ *be such that*

(a) $\overline{u_R}, \underline{u_R} \in L^\infty((0,R)\times(0,T)])$;

(b) $\phi(\overline{u_R}) \in \phi(\overline{u_R}(0,\cdot)) + L^2(0,T;\Omega_R)$, $\phi(\underline{u_R}) \in \phi(\underline{u_R}(0,\cdot)) + L^2(0,T;\Omega_R)$;

(c) $\overline{u_R}_t, \underline{u_R}_t, \phi(\overline{u_R})_{xx}, \phi(\underline{u_R})_{xx} \in L^1((0,R)\times(0,T))$;

(d) $\overline{v_R}, \underline{v_R} \in L^\infty((0,R)\times(0,T))$;

(e) *If* $\varepsilon > 0$, $\phi(\underline{v_R}), \phi(\underline{v_R}) \in L^2(0,T;W^{1,2}(0,R))$, $\overline{v_R}_t, \underline{v_R}_t, \phi(\overline{v_R})_{xx}, \phi(\underline{v_R})_{xx} \in L^1((0,R)\times(0,T))$;

$(\overline{u_R}, \overline{v_R})$, $(\underline{u_R}, \underline{v_R})$ *satisfy*

$$
\begin{array}{ll}
\overline{u_R}_t \geq \phi(\overline{u_R})_{xx} - k\overline{u_R}\overline{v_R}, \quad \underline{u_R}_t \leq \phi(\underline{u_R})_{xx} - k\underline{u_R}\underline{v_R}, & \text{in } (0,R)\times(0,T), \\
\overline{v_R}_t \leq \varepsilon\phi(\overline{v_R})_{xx} - k\overline{u_R}\overline{v_R}, \quad \underline{v_R}_t \geq \varepsilon\phi(\underline{v_R})_{xx} - k\underline{u_R}\underline{v_R}, & \text{in } (0,R)\times(0,T), \\
\overline{u_R}(0,\cdot) \geq \underline{u_R}(0,\cdot), \quad \phi(\overline{v_R})_x(0,\cdot) \leq \phi(\underline{v_R})_x(0,\cdot), & \textit{on } (0,T), \\
\phi(\overline{u_R})_x(R,\cdot) \geq \phi(\underline{u_R})_x(R,\cdot), \quad \phi(\overline{v_R})_x(R,\cdot) \leq \phi(\underline{v_R})_x(R,\cdot), & \textit{on } (0,T), \\
\overline{u_R}(\cdot,0) \geq \underline{u_R}(\cdot,0), \quad \overline{v_R}(0,\cdot) \leq \underline{v_R}(0,\cdot), & \textit{on } (0,R).
\end{array}
$$

Then

$$\overline{u_R} \geq \underline{u_R}, \quad \overline{v_R} \leq \underline{v_R} \quad \textit{in } (0,R)\times(0,T).$$

Proof. Now take a smooth non-decreasing convex function $m^+ : \mathbb{R} \to \mathbb{R}$ with

$$m^+ \geq 0,\ m^+(0) = 0,\ \left(m^+\right)'(0) = 0,\ m^+(r) \equiv 0 \text{ for } r \leq 0,\ m^+(r) = r - \frac{1}{2},$$

for $r > 1$. For $\alpha > 0$, we define the functions

$$m_\alpha^+(r) := \alpha m^+\left(\frac{r}{\alpha}\right),$$

which approximate the positive part of r as $\alpha \to 0$ and $(m_\alpha^+)'(r) \to \operatorname{sgn}^+(r)$ as $\alpha \to 0$. Let $w = \phi(\underline{u_R}) - \phi(\overline{u_R})$ and $z = \phi(\overline{v_R}) - \phi(\underline{v_R})$, we have

$$(\underline{u_R} - \overline{u_R})_t \leq w_{xx} - k(\underline{u_R v_R} - \overline{u_R v_R}), \tag{2.4}$$

$$(\overline{v_R} - \underline{v_R})_t \leq \varepsilon z_{xx} - k(\overline{u_R v_R} - \underline{u_R v_R}). \tag{2.5}$$

Then multiplying (2.4) by $\left(m_\alpha^+\right)'(w)$ and (2.5) by $\left(m_\alpha^+\right)'(z)$ gives

$$
\begin{aligned}
\left(m_\alpha^+\right)'(w)(\underline{u_R} - \overline{u_R})_t &\leq \left(m_\alpha^+\right)'(w)w_{xx} - k\left(m_\alpha^+\right)'(w)(\underline{u_R v_R} - \overline{u_R v_R}), \\
\left(m_\alpha^+\right)'(z)(\overline{v_R} - \underline{v_R})_t &\leq \varepsilon\left(m_\alpha^+\right)'(z)z_{xx} - k\left(m_\alpha^+\right)'(z)(\overline{u_R v_R} - \underline{u_R v_R}),
\end{aligned}
$$

and adding these inequalities gives the following

$$\left(m_\alpha^+\right)'(w)(\underline{u_R}-\overline{u_R})_t+\left(m_\alpha^+\right)'(z)(\overline{v_R}-\underline{v_R})_t$$
$$\le \left(m_\alpha^+\right)'(w)w_{xx}+\varepsilon\left(m_\alpha^+\right)'(z)z_{xx}-k\left[\left(m_\alpha^+\right)'(w)-\left(m_\alpha^+\right)'(z)\right](\underline{u_R}\underline{v_R}-\overline{u_R}\overline{v_R}). \tag{2.6}$$

Now integrating over $(0,R)\times(0,t_0)$, where $t_0\in(0,T]$, gives

$$\int_0^{t_0}\int_0^R\left[\left(m_\alpha^+\right)'(w)w_{xx}+\varepsilon\left(m_\alpha^+\right)'(z)z_{xx}\right]\mathrm{d}x\mathrm{d}t$$
$$=-\int_0^{t_0}\int_0^R\left[\left(m_\alpha^+\right)''(w)|w_x|^2+\varepsilon\left(m_\alpha^+\right)''(z)|z_x|^2\right]\mathrm{d}x\mathrm{d}t\le 0,$$

since $(m_\alpha^+)''(w),(m_\alpha^+)''(z)\ge 0$ because m_α^+ is convex. So (2.6) yields

$$\int_0^{t_0}\int_0^R\left(m_\alpha^+\right)'(w)(\underline{u_R}-\overline{u_R})_t+\left(m_\alpha^+\right)'(z)(\overline{v_R}-\underline{v_R})_t\mathrm{d}x\mathrm{d}t$$
$$\le -k\int_0^{t_0}\int_0^R\left[\left(m_\alpha^+\right)'(w)-\left(m_\alpha^+\right)'(z)\right](\underline{u_R}\underline{v_R}-\overline{u_R}\overline{v_R})\mathrm{d}x\mathrm{d}t.$$

With the nonlinear function ϕ, we need to deal with $(m_\alpha^+)'(w)$ and $(m_\alpha^+)'(s)$ to simplify the left hand side.

Denote $s^+:=\max\{s,0\}$ and

$$\operatorname{sgn}(x)=\begin{cases}-1 & x<0,\\ 0 & x=0\\ 1 & x>0.\end{cases}$$

Now letting $\alpha\to 0$ gives

$$\lim_{\alpha\to0}\left(m_\alpha^+\right)'(w)=\lim_{\alpha\to0}\left(m_\alpha^+\right)'(\phi(\underline{u_R})-\phi(\overline{u_R}))\to\operatorname{sgn}^+(\phi(\underline{u_R})-\phi(\overline{u_R})).$$

Note that $\operatorname{sgn}^+\left[\phi(\underline{u_R})-\phi(\overline{u_R})\right]=\operatorname{sgn}^+(\underline{u_R}-\overline{u_R})$, since ϕ is increasing. By [14, Lemma 7.6], which says that

$$(u_t)^+=\begin{cases}u_t & u>0,\\ 0 & u\le 0,\end{cases} \tag{2.7}$$

we obtain

$$\int_0^{t_0}\int_0^R \left[\mathrm{sgn}^+(\underline{u_R}-\overline{u_R})(\underline{u_R}-\overline{u_R})_t+\mathrm{sgn}^+(\overline{v_R}-\underline{v_R})(\overline{v_R}-\underline{v_R})_t\right]\mathrm{d}x\mathrm{d}t$$
$$=\int_0^{t_0}\int_0^R \left[\left((\underline{u_R}-\overline{u_R})^+\right)_t+\left((\overline{v_R}-\underline{v_R})^+\right)_t\right]\mathrm{d}x\mathrm{d}t$$
$$=\int_0^R \left[(\underline{u_R}-\overline{u_R})^+ + (\overline{v_R}-\underline{v_R})^+\right](x,t_0)\mathrm{d}x-\int_0^R \left[(\underline{u_R}-\overline{u_R})^+ + (\overline{v_R}-\underline{v_R})^+\right](x,0)\mathrm{d}x$$
$$\leq -k\int_0^{t_0}\int_0^R \left[(\mathrm{sgn}w)^+-(\mathrm{sgn}z)^+\right](\underline{u_R v_R}-\overline{u_R v_R})\mathrm{d}x\mathrm{d}t,$$

and the expression

$$\left[(\mathrm{sgn}w)^+-(\mathrm{sgn}z)^+\right](\underline{u_R v_R}-\overline{u_R v_R})\geq 0.$$

Thus

$$\int_0^R \left[(\underline{u_R}-\overline{u_R})^+ + (\overline{v_R}-\underline{v_R})^+\right](x,t_0)\mathrm{d}x$$
$$\leq -k\int_0^{t_0}\int_0^R \left[(\mathrm{sgn}w)^+-(\mathrm{sgn}z)^+\right](\underline{u_R v_R}-\overline{u_R v_R})\mathrm{d}x\mathrm{d}t\leq 0. \tag{2.8}$$

Hence

$$\left[(\underline{u_R}-\overline{u_R})^+ + (\overline{v_R}-\underline{v_R})^+\right](\cdot,t_0)\mathrm{d}x=0 \quad \text{on } (0,R).$$

□

The following corollary is immediately from Lemma 2.2.

Corollary 2.3. *Let $\varepsilon>0$. For given initial data $u_{0,R}, v_{0,R}$, there is at most one solution (u_R,v_R) of (2.1).*

If we take $(\underline{u_R},\underline{v_R})=(0,0)$ and $(\overline{u_R},\overline{v_R})=(u_R,v_R)$, then take $(\underline{u_R},\underline{v_R})=(u_R,v_R)$ and $(\overline{u_R},\overline{v_R})=(U_0,V_0)$ in Lemma 2.2, we can get the following.

Corollary 2.4. *Let (u_R,v_R) be a weak solution of (2.1). Then we have*

$$0\leq u_R(x,t)\leq U_0 \quad \text{and} \quad 0\leq v_R(x,t)\leq V_0, \tag{2.9}$$

for $(x,t)\in(0,R)\times(0,T)$.

We will prove the existence of a weak solution of the appropriate problem (2.1) using an iterative method inspired by [18]. As the first step in the iteration we consider the problem

$$\begin{cases} \left(u_R^{(1)}\right)_t = \phi\left(u_R^{(1)}\right)_{xx} - ku_R^{(1)}V_0, & (x,t) \in (0,R)\times(0,T), \\ u_R^{(1)}(0,t) = U_0, \quad \phi\left(u_R^{(1)}\right)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ u_R^{(1)}(x,0) = u_{0,R}(x), & \text{for} \quad x \in (0,R). \end{cases} \tag{2.10}$$

We will prove the existence and uniqueness of a weak solution $u_R^{(1)}$ in the following and then substitute $u_R^{(1)}$ in the problem

$$\begin{cases} \left(v_R^{(1)}\right)_t = \varepsilon\phi(v_R^{(1)})_{xx} - ku_R^{(1)}v_R^{(1)}, & (x,t) \in (0,R)\times(0,T), \\ \phi\left(v_R^{(1)}\right)_x(0,t) = \phi\left(v_R^{(1)}\right)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ v_R^{(1)}(x,0) = v_{0,R}(x), & \text{for} \quad x \in (0,R), \end{cases} \tag{2.11}$$

and obtain a unique weak solution $v_R^{(1)}$. Our strategy is to replace V_0 in Problem (2.10) by $v_R^{(1)}$ and again we will have a weak solution $u_R^{(2)}$, and so on. In this way, we will obtain sequences $\left\{u_R^{(m)}\right\}$ and $\left\{v_R^{(m)}\right\}$. Finally letting m tend to infinity, we will obtain a solution of Problem (2.1) in the limit.

In order to be able to carry out this procedure, we first introduce a notion of weak solutions for problems of the following type:

$$\begin{cases} u_{Rt} = \phi(u_R)_{xx} - ku_Rp, & (x,t) \in (0,R)\times(0,T), \\ u_R(0,t) = U_0, \quad \phi(u_R)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ u_R(x,0) = u_{0,R}(x), & \text{for} \quad x \in (0,R), \end{cases} \tag{2.12}$$

and

$$\begin{cases} v_{Rt} = \varepsilon\phi(v_R)_{xx} - kv_R q, & (x,t) \in (0,R)\times(0,T), \\ \phi(v_R)_x(0,t) = \phi(v_R)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ v_R(x,0) = v_{0,R}(x), & \text{for} \quad x \in (0,R), \end{cases} \tag{2.13}$$

where $0 \le p \le V_0$ and $0 \le q \le U_0$ almost everywhere in $(0,R)\times(0,T)$.

Definition 2.5. *(I). A function $u_R \in L^\infty((0,R)\times(0,T))$ is called a weak solution of problem (2.12) if*

(i) $\phi(u_R) \in \phi(\hat{u}) + L^2(0,T;\Omega_R)$;

(ii) u_R *satisfies*

$$\begin{aligned} &\int_0^R u_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R u_R\xi_t \mathrm{d}x\mathrm{d}t \\ =&\int_0^T\int_0^R \phi(u_R)_x\xi_x \mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R p \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $\xi \in \mathcal{F}_T^R$.

(II). A function $v_R \in L^\infty((0,R)\times(0,T))$ is called a weak solution of problem (2.13) if

(i) $\phi(v_R) \in L^2(0,T;W^{1,2}(0,R))$;

(ii) v_R *satisfies*

$$\begin{aligned} &\int_0^R v_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R v_R\xi_t \mathrm{d}x\mathrm{d}t \\ =&\int_0^T\int_0^R \varepsilon\phi(v_R)_x\xi_x \mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi v_R q \mathrm{d}x\mathrm{d}t \end{aligned}$$

where $\xi \in \mathcal{F}_T^R$.

Next we will quote the following lemma which is proved in the Appendix in [18]. We will use it to prove the existence of weak solutions of (2.12) and (2.13).

Lemma 2.6. *Let* $\left\{u_R^{(n)}\right\} \subset L^\infty((0,R)\times(0,T))$ *and* $\{\phi_n\} \subset C(\mathbb{R})$ *be sequences with properties*

$$\begin{aligned}
&u_R^{(n)} \rightharpoonup u_R \quad \text{in } L^2((0,R)\times(0,T)),\\
&\phi_n \quad \text{is nondecreasing},\\
&\phi_n \to \phi \quad \text{uniformly on compact subset of } \mathbb{R},\\
&\phi_n(u_R^{(n)}) \to \chi \quad \text{in } L^2((0,R)\times(0,T)),
\end{aligned}$$

then $\chi = \phi(u_R)$.

Now we can prove the following lemma.

Lemma 2.7. *Let* $p, q \in L^\infty((0,R)\times(0,T))$ *be such that* $0 \le p \le V_0$, $0 \le q \le U_0$. *Then problems (2.12) and (2.13) have unique weak solutions* u_R *and* v_R *respectively with the following properties*

$$0 \le u_R \le U_0, \quad 0 \le v_R \le V_0 \quad \text{in } (0,R)\times(0,T).$$

Proof. First we construct the solutions $\left\{u_R^{(n)}\right\}$, $\left\{v_R^{(n)}\right\}$ of sequences of uniformly parabolic problems in which ϕ in (2.12) and (2.13) have been replaced by smooth functions ϕ_n where $\phi_n(U_0) = \phi(U_0)$ and $\phi_n'(u_R^{(n)}) \ge \frac{1}{n}$. Under these assumption on ϕ_n, the equations are parabolic non-degenerate and we may apply standard quasilinear theory to obtain the existence and uniqueness of classical solutions.

We know that for $0 \le u_R^{(n)} \le U_0$, $0 \le v_R^{(n)} \le V_0$ by the usual parabolic comparison principle. Multiplying the equation for $u_R^{(n)}$ by $\phi_n(u_R^{(n)})$ and integrating over $(0,R)\times(0,t_0)$, where $t_0 \in (0,T)$, give

$$\begin{aligned}
&\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)\phi_n\left(u_R^{(n)}\right)_{xx}\mathrm{d}x\mathrm{d}t \\
=&\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)\left(u_R^{(n)}\right)_t\mathrm{d}x\mathrm{d}t + k\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)u_R^{(n)}p\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

gives

$$\begin{aligned}
-\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_x\right|^2\mathrm{d}x\mathrm{d}t =& \int_0^R\left[\Phi\left(u_R^{(n)}\right)(t_0) - \Phi\left(u_R^{(n)}\right)(0)\right]\mathrm{d}x \\
&+ k\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)u_R^{(n)}p\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

then we have

$$\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_x\right|^2\mathrm{d}x\mathrm{d}t \le \left|\int_0^R\left[\Phi\left(u_R^{(n)}\right)(t_0) - \Phi\left(u_R^{(n)}\right)(0)\right]\mathrm{d}x\right| + kU_0V_0\phi(U_0)RT,$$

since $u_R^{(n)} \in L^\infty((0,R)\times(0,T))$, where $\Phi'(s) = \phi(s)$.

To prove the L^2 bound of $\phi_n\left(u_R^{(n)}\right)_t$, we multiply the equation for $u_R^{(n)}$ by $\phi_n\left(u_R^{(n)}\right)_t$ and integrate over $(0,R)\times(0,t_0)$, then

$$\begin{aligned}
\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)_t\left(u_R^{(n)}\right)_t\mathrm{d}x\mathrm{d}t =& \int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)_t\phi_n\left(u_R^{(n)}\right)_{xx}\mathrm{d}x\mathrm{d}t \\
&- k\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)_t u_R^{(n)}p\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Denoting $N = \phi'(U_0)$, we get

$$\begin{aligned}
&\frac{1}{N}\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_t\right|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^R\left|\phi_n\left(u_R^{(n)}\right)_x(t_0)\right|^2\mathrm{d}x \\
\le&\frac{1}{2}\int_0^R\left|\phi_n\left(u_R^{(n)}\right)_x(0)\right|^2\mathrm{d}x - k\int_0^{t_0}\int_0^R \phi_n\left(u_R^{(n)}\right)_t u_R^{(n)}p\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

because ϕ' is increasing, so $\phi'(s) \le N$ for $s \in [0,U_0]$.

We now estimate the last term by the Cauchy-Schwarz Inequality, which yields

$$
\begin{aligned}
&\frac{1}{N}\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_t\right|^2 \mathrm{d}x\mathrm{d}t\\
\leq&\frac{1}{2}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_x(0)\right|^2 \mathrm{d}x + kC\left[\int_0^{t_0}\int_0^R \left(\phi_n\left(u_R^{(n)}\right)_t\right)^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}},
\end{aligned}
$$

where C is a positive constant since $0 \leq u_R^{(n)} \leq U_0$ and $0 \leq p \leq V_0$.

This implies that

$$
\begin{aligned}
&\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_t\right|^2 \mathrm{d}x\mathrm{d}t && (2.14)\\
\leq&C_1\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_x(0)\right|^2 \mathrm{d}x + kC_2\left[\int_0^{t_0}\int_0^R \left(\phi_n\left(u_R^{(n)}\right)_t\right)^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}}, && (2.15)
\end{aligned}
$$

where C_1 and C_2 are positive constants. With

$$
X = \left[\int_0^{t_0}\int_0^R \left(\phi_n\left(u_R^{(n)}\right)_t\right)^2\right]^{\frac{1}{2}},
$$

we can write (2.15) as

$$
X^2 \leq C_1\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_x(0)\right|^2 \mathrm{d}x + kC_2X,
$$

from which we conclude that

$$
\int_0^{t_0}\int_0^R \left|\phi_n\left(u_R^{(n)}\right)_t\right|^2 \mathrm{d}x\mathrm{d}t \leq C_3,
$$

where C_3 is a positive constant, since $\phi_n\left(u_R^{(n)}\right)_x(0) \in L^1((0,R))$.

Then we know that $\phi_n\left(u_R^{(n)}\right)$ is bounded in $L^\infty((0,R)\times(0,T))$, $\phi_n\left(u_R^{(n)}\right)_x$ is bounded in $L^2((0,R)\times(0,T))$ and $\phi_n\left(u_R^{(n)}\right)_t$ is bounded in $L^2((0,R)\times(0,T))$. By [13, p.170], which says that $W^{1,1}(\Omega) \subset BV(\Omega)$, we therefore have $\phi_n\left(u_R^{(n)}\right)$ is bounded in $BV((0,R)\times(0,T))$ and there exists a subsequence $\left\{u_R^{(n_j)}\right\}$ and a function $\chi_1 \in BV((0,R)\times(0,T))$ such that

$$
\phi_{n_j}\left(u_R^{(n_j)}\right) \to \chi_1 \quad \text{in } L^1((0,R)\times(0,T)) \text{ as } j\to\infty.
$$

As $\varepsilon > 0$, it follows similarly for v that there exists a subsequence $\left\{v_R^{(n_j)}\right\}$ and a function $\chi_2 \in BV((0,R)\times(0,T))$ such that

$$\phi_{n_j}\left(v_R^{(n_j)}\right) \to \chi_2 \quad \text{in } L^1((0,R)\times(0,T)) \text{ as } j\to\infty.$$

We may choose these sequences such that

$$u_R^{(n_j)} \rightharpoonup u_R, \quad v_R^{(n_j)} \rightharpoonup v_R \quad \text{in } L^2((0,R)\times(0,T)),$$

and sequence $\{\phi_n\}$ such that $\phi_n \to \phi$ uniformly. By Lemma 2.6 we have $\chi_1 = \phi(u_R)$, $\chi_2 = \phi(v_R)$. We know that $\phi_n\left(u_R^{(n)}\right) - \phi(U_0)$ is bounded in $L^2(0,T;\Omega_R)$ and $\phi_n\left(v_R^{(n)}\right)$ is bounded in $L^2(0,T;W^{1,2}(0,R))$, so there are subsequences, again denote by $\left\{u_R^{(n_j)}\right\}$ and $\left\{v_R^{(n_j)}\right\}$ such that

$$\begin{aligned}\phi_{n_j}\left(u_R^{(n_j)}\right) - \phi(U_0) \rightharpoonup \phi(u_R) - \phi(U_0) \quad &\text{in } L^2(0,T;\Omega_R),\\ \phi_{n_j}\left(v_R^{(n_j)}\right) \rightharpoonup \phi(v_R) \quad &\text{in } L^2((0,R)\times(0,T)).\end{aligned}$$

By a standard limiting argument we can show that u_R is a weak solution of Problem (2.12) and v_R is a weak solution of Problem (2.13).

The uniqueness is shown in a way similar to the proof of Lemma 2.2. □

We now return to problems (2.10) and (2.11). From Lemma 2.7 we immediately deduce that $u_R^{(1)}$ and $v_R^{(1)}$ are weak solutions of (2.10) and (2.11). We then define the sequences $\left\{u_R^{(m)}\right\}$ and $\left\{v_R^{(m)}\right\}$ inductively as following, let $u_R^{(m)}$ be the weak solution of the problem

$$\begin{cases} u_{Rt}^{(m)} = \phi\left(u_R^{(m)}\right)_{xx} - k u_R^{(m)} v_R^{(m-1)}, & (x,t)\in(0,R)\times(0,T),\\ u_R^{(m)}(0,t) = U_0, \quad \phi\left(u_R^{(m)}\right)_x (R,t) = 0, & \text{for} \quad t\in(0,T),\\ u_R^{(m)}(x,0) = u_{0,R}(x), & \text{for} \quad x\in(0,R), \end{cases} \tag{2.16}$$

let $v_R^{(m)}$ be the weak solution of the problem

$$\begin{cases} v_{Rt}^{(m)} = \varepsilon\phi(v_R^{(m)})_{xx} - ku_R^{(m)}v_R^{(m)}, & (x,t) \in (0,R)\times(0,T), \\ \phi\left(v_R^{(m)}\right)_x(0,t) = \phi\left(v_R^{(m)}\right)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ v_R^{(m)}(x,0) = v_{0,R}(x), & \text{for} \quad x \in (0,R). \end{cases} \tag{2.17}$$

Then $u_R^{(m)}$ and $v_R^{(m)}$ satisfy

(i) $\phi\left(u_R^{(m)}\right) \in \phi(\hat{u}) + L^2(0,T;\Omega_R)$, and

$$\begin{aligned} &\int_0^R u_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R u_R^{(m)}\xi_t\mathrm{d}x\mathrm{d}t \\ =& \int_0^T\int_0^R \phi\left(u_R^{(m)}\right)_x \xi_x\mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R^{(m)}v_R^{(m-1)}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.18}$$

where $\xi \in \mathcal{F}_T^R$.

(ii) $\phi\left(v_R^{(m)}\right) \in L^2(0,T;W^{1,2}(0,R))$, and

$$\begin{aligned} &\int_0^R v_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R v_R^{(m)}\xi_t\mathrm{d}x\mathrm{d}t \\ =& \int_0^T\int_0^R \varepsilon\phi\left(v_R^{(m)}\right)_x \xi_x\mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R^{(m)}v_R^{(m)}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.19}$$

where $\xi \in \mathcal{F}_T^R$.

In the following, we prove the monotone dependence of $u_R^{(m)}, v_R^{(m)}$ on m.

Lemma 2.8. *The problem (2.16) and (2.17) have unique solutions with the following properties*

(i) $u_R^{(m)}$ *and* $v_R^{(m)}$ *are weak solutions of problems (2.16) and (2.17)*;

(ii) $0 \le u_R^{(m)} \le u_R^{(m+1)} \le U_0, \qquad 0 \le v_R^{(m+1)} \le v_R^{(m)} \le V_0.$

Proof. The proof proceeds by induction. We first note that $u_R^{(1)}$ and $u_R^{(2)}$ satisfy the equations

$$\begin{aligned} u_{Rt}^{(1)} &= \phi\left(u_R^{(1)}\right)_{xx} - k u_R^{(1)} V_0, \\ u_{Rt}^{(2)} &= \phi\left(u_R^{(2)}\right)_{xx} - k u_R^{(2)} v_R^{(1)}, \end{aligned}$$

almost everywhere in $(0, R) \times (0, T)$.

Since $u_R^{(1)}, u_R^{(2)}$ satisfy identical initial and boundary conditions, $u_R^{(1)}$ is a subsolution for (2.16) with $m = 2$, since $v_R^{(1)} \leq V_0$, which implies $u_R^{(2)} \geq u_R^{(1)}$. Now consider $v_R^{(1)}$ and $v_R^{(2)}$

$$\begin{aligned} v_{Rt}^{(1)} &= \varepsilon\phi\left(v_R^{(1)}\right)_{xx} - k u_R^{(1)} v_R^{(1)}, \\ v_{Rt}^{(2)} &= \varepsilon\phi\left(v_R^{(2)}\right)_{xx} - k u_R^{(2)} v_R^{(2)}. \end{aligned}$$

Since $v_R^{(1)}, v_R^{(2)}$ satisfy identical initial and boundary conditions, $v_R^{(2)}$ is a subsolution for (2.17) as $m = 2$, which implies $v_R^{(1)} \geq v_R^{(2)}$. The proof of monotone dependence of $u_R^{(m)}$ and $v_R^{(m)}$ of m for large values of m is similar. □

We can now establish the existence of a weak solution of Problem (2.1) when ε is strictly positive.

Theorem 2.9. *There exists a unique weak solution (u_R, v_R) of Problem (2.1) such that*

$$0 \leq u_R \leq U_0 \quad \text{and} \quad 0 \leq v_R \leq V_0.$$

Proof. Lemma 2.8 implies that the functions $u_R^{(m)}$ and $v_R^{(m)}$ tend (pointwise) to functions u_R, v_R as m tends to infinity. By the proof of Lemma 2.7 we conclude there are subsequences $\left\{u_R^{(m_j)}\right\}$ and $\left\{v_R^{(m_j)}\right\}$ such that

$$\phi\left(u_R^{(m_j)}\right) - \phi(U_0) \rightharpoonup \phi(u_R) - \phi(U_0) \quad \text{weakly in } L^2(0, T; \Omega_R),$$

$$\phi\left(v_R^{(m_j)}\right) \rightharpoonup \phi(v_R) \quad \text{weakly in } L^2(0,T;W^{1,2}(0,R)).$$

Then, by the Dominated Convergence Theorem and passing to the limits as (2.18) leads to

$$\begin{aligned}
&\int_0^R u_{0,R}\xi(x,0)\mathrm{d}x + \int_0^T\int_0^R u_R\xi_t \mathrm{d}x\mathrm{d}t \\
=& \int_0^T\int_0^R \phi(u_R)_x\xi_x \mathrm{d}x\mathrm{d}t + k\int_0^T\int_0^R \xi u_R v_R \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\xi \in \mathcal{F}_T^R$. We can readily show u_R is a weak solution of (2.12) with $p = v_R$. From DiBenedetto [11, Theorem 7.1], we conclude that $u_R \in C([0,R]\times[0,T])$. Similarly, we know that v_R is a weak solution of problem (2.13) with $q = u_R$ and we can conclude also that $v_R \in C([0,R]\times[0,T])$. It follows from Lemma 2.2 that (u_R, v_R) is the unique weak solution of problem (2.1). □

Next, we will prove the existence of weak solution of (1.4) with $\varepsilon > 0$ by looking at (u_R, v_R) in the limit $R \to \infty$. First, we prove some preliminary estimates. In the following, $C(L)$ denotes some L-dependent constant which varies according to context.

Lemma 2.10. *Suppose $\varepsilon > 0$ and $L > 0$. Then there exists a constant $C(L)$ independent of k such that if $R > L+1$, then*

$$k\int_0^T\int_0^{L+1} u_R v_R \mathrm{d}x\mathrm{d}t \le C(L). \tag{2.20}$$

Proof. Introducing a cut-off function $\varphi^1 \in C^\infty(\mathbb{R}^+)$ such that $0 \le \varphi^1(x) \le 1$ for all $x \in \mathbb{R}^+$, $\varphi^1(0) = \varphi^1_x(0) = 0$,

$$\varphi^1(x) = \begin{cases} 1 & x \in [1,2], \\ 0 & x \ge 3. \end{cases}$$

Then given $L \geq 2$, define the family of cut-off functions $\varphi^L \in C^\infty(\mathbb{R}^+)$ by

$$\varphi^L(x) = \begin{cases} \varphi^1(x) & x \in [0,1], \\ 1 & x \in [1,L], \\ \varphi^1(x+2-L) & x \geq L. \end{cases} \tag{2.21}$$

Note that $0 \leq \varphi^L \leq 1$ for all L, and φ^L_x, φ^L_{xx} are bounded in $L^\infty(\mathbb{R}^+)$ independently of L.

Multiplying the equation for u_R by φ^L and integrating over $(0,R)\times(0,T)$ gives that

$$k\int_0^T\int_0^{L+1} u_R v_R \varphi^L \mathrm{d}x\mathrm{d}t = \int_0^T\int_0^{L+1} \phi(u_R)\varphi^L_{xx}\mathrm{d}x\mathrm{d}t + \int_0^{L+1}\varphi^L u_{0,R}\mathrm{d}x$$
$$- \int_0^{L+1}\varphi^L u_R(x,T)\mathrm{d}x.$$

The fact that $0 \leq u_R \leq U_0$, together with the Lebesgue's Monotone Convergence Theorem, yield (2.20). $\square$

Lemma 2.11. *Suppose $\varepsilon > 0$. Then for each $L \geq 1$, $\phi(u_R), \phi(v_R)$ are bounded in $L^2\big(0,T;W^{1,2}(0,L)\big)$ independently of k and R.*

Proof. Now we introduce a cut-off function $\psi^1 \in C^\infty(\mathbb{R}^+)$ such that $0 \leq \psi^1 \leq 1$ for $x \in \mathbb{R}^+$

$$\psi^1 = \begin{cases} 1 & x \leq 1, \\ 0 & x \geq 2. \end{cases}$$

Then given $L \geq 1$, define the family of cut-off functions $\psi^L \in C^\infty(\mathbb{R}^+)$ by

$$\psi^L = \begin{cases} 1 & x \leq L, \\ \psi^1(x+1-L) & x \geq L. \end{cases}$$

Clearly ψ^L, ψ^L_x and ψ^L_{xx} are bounded in $L^\infty(\mathbb{R}^+)$ independently of L. Suppose that $R > L+1$. Then multiplying the equation for u_R by $\big[\phi(u_R)-\phi(U_0)\big]\psi^L$ and integrating over $(0,R)\times(0,T)$ give

$$\begin{aligned}
&\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]\psi^L u_{Rt}\mathrm{d}x\mathrm{d}t\\
&=-\int_0^T\int_0^{L+1}|\phi(u_R)_x|^2\psi^L\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]^2\psi^L_{xx}\mathrm{d}x\mathrm{d}t\\
&\quad-k\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]\psi^L u_R v_R\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

Now let $F=\displaystyle\int_0^{u_R}\phi(s)\mathrm{d}s$, we have

$$\begin{aligned}
\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]\psi^L u_{Rt}\mathrm{d}x\mathrm{d}t=&\int_0^{L+1}\big[F(x,T)-F(x,0)\big]\psi^L\mathrm{d}x\\
&+\int_0^{L+1}\phi(U_0)\big(u_{0,R}-u_R(x,T)\big)\psi^L\mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^T\int_0^{L+1}|\phi(u_R)_x|^2\psi^L\mathrm{d}x\mathrm{d}t=&-\int_0^{L+1}\phi(U_0)\big(u_{0,R}-u_R(x,T)\big)\psi^L\mathrm{d}x\\
&-\int_0^{L+1}\big[F(x,T)-F(x,0)\big]\psi^L\mathrm{d}x\\
&+\frac{1}{2}\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]^2\psi^L_{xx}\mathrm{d}x\mathrm{d}t\\
&-k\int_0^T\int_0^{L+1}\big[\phi(u_R)-\phi(U_0)\big]\psi^L u_R v_R\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By (2.9) and the fact that ϕ is increasing with respect to u_R, we know that there exists some C such that

$$F=\int_0^{u_R}\phi(s)\mathrm{d}s\le C,$$

We know $\phi(u_R)-\phi(U_0)\in L^\infty((0,R)\times(0,T))$, then Lemma 2.10 yields

$$\int_0^T\int_0^{L+1}|\phi(u_R)_x|^2\mathrm{d}x\mathrm{d}t\le C, \tag{2.22}$$

independently of k and R. If $\varepsilon > 0$, the estimate for $\phi(v_R)_x$ can be proved likewise, using the equation for v_R. $\square$

In order to prove that the sets $\{u_R\}_{R>0}, \{v_R\}_{R>0}$ are each relatively compact in $L^2_{loc}(\mathbb{R}^+ \times (0,T))$, we now prove estimates of space and time translates of u_R, v_R.

It is convenient to introduce a shorthand notation for space and time translates. Given a function h, let

$$S_\delta h(x,t) := h(x+\delta,t), \quad T_\tau h(x,t) := h(x,t+\tau), \tag{2.23}$$

for all (x,t) in a suitable space-time domain and appropriate δ and τ.

Lemma 2.12. *Suppose $\varepsilon > 0$. Then for each $L > 0$ and $r \in (0,1)$, there exists a constant $C(L)$, independent of k and δ, such that*

$$\int_0^T \int_r^{L+1} |\phi(S_\delta u_R) - \phi(u_R)|^2 \mathrm{d}x\mathrm{d}t \leq C(L)|\delta|^2,$$
$$\int_0^T \int_r^{L+1} |\phi(S_\delta v_R) - \phi(v_R)|^2 \mathrm{d}x\mathrm{d}t \leq C(L)|\delta|^2,$$

for all $\delta \in \mathbb{R}$, $|\delta| \leq r$.

Proof. As a result of the gradient bounds in Lemma 2.11, this can be proved by adapting the proof of Lemma 2.6 in [9]. Indeed

$$\begin{aligned}
&\int_0^T \int_r^{L+1} |\phi(u_R)(x+\delta,t) - \phi(u_R)(x,t)|^2 \mathrm{d}x\mathrm{d}t \\
=&\int_0^T \int_r^{L+1} \left| \int_0^1 \phi(u_R)_x(x+\theta\delta,t) \cdot \delta \mathrm{d}\theta \right|^2 \mathrm{d}x\mathrm{d}t \\
\leq&|\delta|^2 \int_0^1 \int_0^T \int_r^{L+1} \left|\phi(u_R)_x(x+\theta\delta,t)\right|^2 \mathrm{d}x\mathrm{d}t\mathrm{d}\theta \\
\leq&|\delta|^2 \int_0^T \int_0^{L+2} \left|\phi(u_R)_x(x,t)\right|^2 \mathrm{d}x\mathrm{d}t \\
\leq&C|\delta|^2.
\end{aligned}$$

An analogous estimate for v_R can be obtained using similar arguments. $\square$

Lemma 2.13. *Suppose $\varepsilon > 0$. Then for each $L > 0$, there exists a constant $C(L)$ independent of k and $\tau \in (0,T)$ such that*

$$\int_0^{T-\tau}\int_0^{L+1} |\phi(T_\tau u_R) - \phi(u_R)|^2 \mathrm{d}x\mathrm{d}t \le \tau C(L),$$
$$\int_0^{T-\tau}\int_0^{L+1} |\phi(T_\tau v_R) - \phi(v_R)|^2 \mathrm{d}x\mathrm{d}t \le \tau C(L).$$

Proof. The proof takes the advantage of [10, Lemma 2.16], see also [9, Lemma 3]. Since we have nonlinear diffusion terms, we also need to deal with the nonlinearity ϕ. Let ψ^L be as in the proof of Lemma 2.11. Then it follows using the Mean Value Theorem that

$$\begin{aligned}
&\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right|^2 \mathrm{d}x\mathrm{d}t \\
=&\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| \phi'(\rho) \left|T_\tau u_R - u_R\right| \mathrm{d}x\mathrm{d}t \\
\le& N\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| \left|T_\tau u_R - u_R\right| \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\rho \in [0, U_0]$ and $N = \phi'(U_0)$ such that $\phi'(s) \le N$ for all $s \in [0, U_0]$, since ϕ' is increasing. Then we have

$$\begin{aligned}
&\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right|^2 \mathrm{d}x\mathrm{d}t \\
\le& N\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| \left[\int_0^{\tau} (u_R)_s(x, t+s)\mathrm{d}s\right] \mathrm{d}x\mathrm{d}t \\
=& N\int_0^{\tau}\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi\left(T_\tau u_R\right) - \phi(u_R)\right| \phi(u_R)_{xx}(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s \\
&- Nk\int_0^{\tau}\int_0^{T-\tau}\int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| u_R(x, t+s) v_R(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s \\
=& I_1 + I_2 + I_3,
\end{aligned}$$

with

$$I_1 := -N \int_0^\tau \int_0^{T-\tau} \int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right|_x \phi(u_R)_x(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s,$$

$$I_2 := -N \int_0^\tau \int_0^{T-\tau} \int_0^{L+1} \psi_x^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| \phi(u_R)_x(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s,$$

$$I_3 := -Nk \int_0^\tau \int_0^{T-\tau} \int_0^{L+1} \psi^L \left|\phi(T_\tau u_R) - \phi(u_R)\right| u_R(x, t+s)v_R(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s.$$

I_1 can be split into two terms and by using Cauchy-Schwarz inequality and using the property $\psi^L \leq 1$ yield

$$\begin{aligned}
|I_1| = &\left| -N \int_0^\tau \int_0^{T-\tau} \int_0^{L+1} \psi^L \phi(T_\tau u_R)_x \phi(u_R)_x(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s \right. \\
&\left. +N \int_0^\tau \int_0^{T-\tau} \int_0^{L+1} \psi^L \phi(u_R)_x \phi(u_R)_x(x, t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s \right| \\
\leq &2\tau N \left\{ \int_0^T \int_0^{L+1} \left|\phi(u_R(x,t))_x\right|^2 \mathrm{d}x\mathrm{d}t \right\}^{\frac{1}{2}}.
\end{aligned}$$

which is bounded by Lemma 2.11.

By (2.9) and the Cauchy-Schwarz inequality, there exist C independent of k such that

$$\begin{aligned}
|I_2| = &\sup|\psi_x^L| NC \int_0^\tau \int_0^{T-\tau} \int_L^{L+1} \left|\phi(u_R)_x(x, t+s)\right| \mathrm{d}x\mathrm{d}t\mathrm{d}s \\
\leq &\sup|\psi_x^L| NC\tau \left\{ \int_0^{T-\tau} \int_L^{L+1} \left|\phi(u_R)_x(x, t+s)\right|^2 \mathrm{d}x\mathrm{d}t \right\}^{\frac{1}{2}} \mathrm{d}s,
\end{aligned}$$

which is bounded by (2.22) and the fact that $\sup|\psi_x^L|$ is bounded independent of L.

The last term is easier to handle, by (2.9) we get

$$|I_3| \leq 2M\tau N \int_0^T \int_0^{L+1} k u_R v_R \mathrm{d}x\mathrm{d}t,$$

which is bounded by Lemma 2.10. An analogous estimate for v_R can be obtained by using similar arguments. $\square$

We can now establish the existence of a weak solution of the original problem (1.4) of S_T when $\varepsilon > 0$. Now with

$$\Omega_J := \left\{\alpha \in W^{1,2}((0,J)) |\ \alpha = 0 \text{ at } x = 0\right\},$$

and define

$$\mathcal{F}_T := \big\{\xi \in C^1(S_T):\ \xi(0,t) = \xi(\cdot,T) = 0 \text{ for } t \in (0,T) \text{ and } \operatorname{supp}\xi \subset [0,J]\times[0,T] \text{ for some } J > 0\big\}.$$

Theorem 2.14. *Let $\varepsilon > 0$. Given $k > 0$, there exists a weak solution $(u^k, v^k) \in (L^\infty(S_T))^2$ of (1.4) such that for each $J > 0$,*

(i) $\phi(u^k) \in \phi(\hat{u}) + L^2(0,T;\Omega_J), \quad \phi(v^k) \in L^2(0,T;W^{1,2}((0,J)))$

(ii) (u^k, v^k) *satisfies*

$$\int_{\mathbb{R}^+} u_0^k \xi(x,0)\mathrm{d}x + \iint_{S_T} u^k \xi_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} \phi(u^k)_x \xi_x \mathrm{d}x\mathrm{d}t + k\iint_{S_T} \xi u^k v^k \mathrm{d}x\mathrm{d}t,$$
$$\int_{\mathbb{R}^+} v_0^k \xi(x,0)\mathrm{d}x + \iint_{S_T} v^k \xi_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} \varepsilon\phi(v^k)_x \xi_x \mathrm{d}x\mathrm{d}t + k\iint_{S_T} \xi u^k v^k \mathrm{d}x\mathrm{d}t,$$

where $\xi \in \mathcal{F}_T$.

Proof. Let $u_{0,R}, v_{0,R}$ be as in the formulation of problem (2.2) and note that as $R \to \infty$, $u_{0,R} \to u_0^k$, $v_{0,R} \to v_0^k$ in $C^1_{loc}(\mathbb{R}^+)$. Then given $R_n \to \infty$, it follows from the Fréchet-Kolmogorov Theorem (see, for example, [3, Corollary 4.27]) and (2.9), Lemma 2.12 and 2.13, that there exist subsequences $\{R_{n_j}\}$ and functions $u^k \in L^\infty(S_T)$ and $v^k \in L^\infty(S_T)$ such that

$$u_{R_{n_j}} \to u^k, \quad v_{R_{n_j}} \to v^k \text{ strongly in } L^2_{loc}(S_T) \text{ and a.e. in } S_T$$

as $j \to \infty$. Now we know that $\phi(u_{R_{n_j}}) - \phi(U_0)$ is bounded in $L^2(0,T;\Omega_J)$ and $\phi(v_{R_{n_j}})$ is bounded in $L^2(0,T;W^{1,2}(0,J))$ by Lemma 2.11, then we have as $j \to \infty$

$$\phi(u_{R_{n_j}}) - \phi(U_0) \rightharpoonup \phi(u^k) - \phi(U_0) \quad \text{in } L^2(0,T;\Omega_J),$$

$$\phi(v_{R_{n_j}}) \rightharpoonup \phi(v^k) \quad \text{in } L^2(0,T;W^{1,2}((0,J))).$$

By the Dominated Convergence Theorem we can then easily pass to the limit in the weak form of (1.4). $\square$

We use the following comparison principle theorem for (1.4) to show the uniqueness of the weak solution of (1.4). Note that this result covers both the case $\varepsilon > 0$ and the case $\varepsilon = 0$.

Lemma 2.15. *Let $\varepsilon \geq 0$ and $(\overline{u},\overline{v})$, $(\underline{u},\underline{v})$ be such that*

(a) $\overline{u}, \underline{u} \in L^\infty(S_T)$;

(b) $\phi(\overline{u}) \in \phi(\overline{u}(0,\cdot)) + L^2(0,T;\Omega_J)$, $\phi(\underline{u}) \in \phi(\underline{u}(0,\cdot)) + L^2(0,T;\Omega_J)$;

(c) $\overline{u}_t, \underline{u}_t, \phi(\overline{u})_{xx}, \phi(\underline{u})_{xx} \in L^1(S_T)$;

(d) $\overline{v}, \underline{v} \in L^\infty(S_T)$, $\overline{v}_t, \underline{v}_t \in L^1(S_T)$;

(e) *If* $\varepsilon > 0$, $\phi(\overline{v}), \phi(\underline{v}) \in L^2(0,T;W^{1,2}((0,J)))$, $\phi(\overline{v})_{xx}, \phi(\underline{v})_{xx} \in L^1(S_T)$;

and $(\bar{u},\bar{v})$, $(\underline{u},\underline{v})$ satisfy

$$\begin{aligned}
&\overline{u}_t \geq \phi(\overline{u})_{xx} - k\overline{u}\overline{v}, \quad \underline{u}_t \leq \phi(\underline{u})_{xx} - k\underline{u}\underline{v}, && \textit{in } S_T,\\
&\overline{v}_t \leq \varepsilon\phi(\overline{v})_{xx} - k\overline{u}\overline{v}, \quad \underline{v}_t \geq \varepsilon\phi(\underline{v})_{xx} - k\underline{u}\underline{v}, && \textit{in } S_T,\\
&\overline{u}(0,\cdot) \geq \underline{u}(0,\cdot), \quad \varepsilon\phi(\overline{v})_x(0,\cdot) = \varepsilon\phi(\underline{v})_x(0,\cdot) = 0, && \textit{on } (0,T),\\
&\overline{u}(\cdot,0) \geq \underline{u}(\cdot,0), \quad \overline{v}(0,\cdot) \leq \underline{v}(0,\cdot), && \textit{on } \mathbb{R}^+.
\end{aligned}$$

Then

$$\overline{u} \geq \underline{u}, \quad \overline{v} \leq \underline{v} \quad \textit{in } S_T.$$

Proof. The proof are similar to the proof of Lemma 2.2. Now take function m^+ as in the proof of Lemma 2.2 and let $w = \phi(\underline{u}) - \phi(\overline{u})$ and $z = \phi(\overline{v}) - \phi(\underline{v})$, we have

$$(\underline{u} - \overline{u})_t \leq w_{xx} - k(\underline{uv} - \overline{uv}), \tag{2.24}$$

$$(\overline{v} - \underline{v})_t \leq \varepsilon z_{xx} - k(\overline{uv} - \underline{uv}). \tag{2.25}$$

Let ψ^L be as in the proof of Lemma 2.11. Multiplying (2.24) by $(m_\alpha^+)'(w)\psi^L$ and (2.25) by $(m_\alpha^+)'(z)\psi^L$ and then adding these inequalities yield

$$\begin{aligned} &\left(m_\alpha^+\right)'(w)\psi^L(\underline{u} - \overline{u})_t + \left(m_\alpha^+\right)'(z)\psi^L(\overline{v} - \underline{v})_t \\ \leq &\left(m_\alpha^+\right)'(w)\psi^L w_{xx} + \varepsilon\left(m_\alpha^+\right)'(z)\psi^L z_{xx} - k\psi^L\left[\left(m_\alpha^+\right)'(w) - \left(m_\alpha^+\right)'(z)\right](\underline{uv} - \overline{uv}), \end{aligned} \tag{2.26}$$

integrating over $\mathbb{R} \times (0, t_0)$, where $t_0 \in (0, T]$, gives the following

$$\begin{aligned} &\int_0^{t_0}\int_{\mathbb{R}^+} \psi^L\left[\left(m_\alpha^+\right)'(w)w_{xx} + \varepsilon\left(m_\alpha^+\right)'(z)z_{xx}\right]\mathrm{d}x\mathrm{d}t \\ \leq &-\int_0^{t_0}\int_{\mathbb{R}^+}\psi_x^L\left[m_\alpha^+(w)\right]_x\mathrm{d}x\mathrm{d}t - \varepsilon\int_0^{t_0}\int_{\mathbb{R}^+}\psi_x^L\left[m_\alpha^+(z)\right]_x\mathrm{d}x\mathrm{d}t \\ = &\int_0^{t_0}\int_{\mathbb{R}^+}\psi_{xx}^L\left[m_\alpha^+(w) + \varepsilon m_\alpha^+(z)\right]\mathrm{d}x\mathrm{d}t, \end{aligned}$$

since $(m_\alpha^+)''(w), (m_\alpha^+)''(z) \geq 0$ because m_α^+ is convex. So (2.26) yields

$$\begin{aligned} &\int_0^{t_0}\int_{\mathbb{R}^+}\left(m_\alpha^+\right)'(w)(\underline{u} - \overline{u})_t + \left(m_\alpha^+\right)'(z)(\overline{v} - \underline{v})_t\mathrm{d}x\mathrm{d}t \\ \leq &\int_0^{t_0}\int_{\mathbb{R}^+}\psi_{xx}^L\left[m_\alpha^+(w) + \varepsilon m_\alpha^+(z)\right]\mathrm{d}x\mathrm{d}t \\ &-k\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L\left[\left(m_\alpha^+\right)'(w) - \left(m_\alpha^+\right)'(z)\right](\underline{uv} - \overline{uv})\mathrm{d}x\mathrm{d}t, \end{aligned}$$

and letting $\alpha \to 0$ gives

$$\lim_{\alpha\to 0}\left(m_\alpha^+\right)'(w) = \lim_{\alpha\to 0}\left(m_\alpha^+\right)'(\phi(\underline{u}) - \phi(\overline{u})) \to \operatorname{sgn}^+(\phi(\underline{u}) - \phi(\overline{u})).$$

Clearly

$$\mathrm{sgn}^+\left[\phi(\underline{u})-\phi(\overline{u})\right]=\mathrm{sgn}^+(\underline{u}-\overline{u}),$$

since ϕ is increasing. Denoting $u^+:=\max\{u,0\}$. [14, Lemma 7.6] gives the following

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L\left[\left((\underline{u}-\overline{u})^+\right)_t+\left((\overline{v}-\underline{v})^+\right)_t\right]\mathrm{d}x\mathrm{d}t\\
=&\int_{\mathbb{R}^+}\psi^L\left[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+\right](x,t_0)\mathrm{d}x\\
&-\int_{\mathbb{R}^+}\psi^L\left[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+\right](x,0)\mathrm{d}x\\
\le&\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L_{xx}(w^++\varepsilon z^+)\mathrm{d}x\mathrm{d}t\\
&-k\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L\left[(\mathrm{sgn}w)^+-(\mathrm{sgn}z)^+\right](\underline{uv}-\overline{uv})\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and the expression

$$\left[(\mathrm{sgn}w)^+-(\mathrm{sgn}z)^+\right](\underline{uv}-\overline{uv})\ge 0.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^+}\psi^L\left[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+\right](x,t_0)\mathrm{d}x\\
\le&\int_{\mathbb{R}^+}\psi^L\left[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+\right](x,0)\mathrm{d}x+\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L_{xx}(w^++\varepsilon z^+)\mathrm{d}x\mathrm{d}t\\
\le&\int_0^{t_0}\int_{\mathbb{R}^+}\psi^L_{xx}\left\{\left[\phi(\underline{u})-\phi(\overline{u})\right]^++\varepsilon\left[\phi(\overline{v})-\phi(\underline{v})\right]^+\right\}\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{2.27}$$

which is bounded independently of L and t_0 by the definition of ψ^L and in particular, $\psi^L_{xx}\neq 0$ only if $x\in[L,L+1]$. Now using Lebesgue's Monotone Convergence Theorem we deduce that $[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+]\in L^\infty\left(0,T;L^1(\mathbb{R}^+)\right)$, thus (2.27) tends to 0 as $L\to\infty$. Hence

$$\left[(\underline{u}-\overline{u})^++(\overline{v}-\underline{v})^+\right](\cdot,t_0)=0\quad\text{on }\mathbb{R}^+.$$

□

The following corollary is immediately from Lemma 2.2.

Corollary 2.16. *Let $\varepsilon \geq 0$. For given initial data u_0^k, v_0^k, there is at most one solution (u^k, v^k) of (1.4).*

If we take $(\underline{u}, \underline{v}) = (0, 0)$ and $(\overline{u}, \overline{v}) = (u^k, v^k)$, then take $(\underline{u}, \underline{v}) = (u^k, v^k)$ and $(\overline{u}, \overline{v}) = (U_0, V_0)$ in Lemma 2.2, we can get the following.

Corollary 2.17. *Let $\varepsilon \geq 0$ and (u^k, v^k) be a weak solution of (1.4). Then for given $k > 0$, we have*

$$0 \leq u^k(x,t) \leq U_0 \quad \text{and} \quad 0 \leq v^k(x,t) \leq V_0 \quad \text{for } (x,t) \in S_T. \tag{2.28}$$

The existence of weak solutions of (1.4) when $\varepsilon = 0$ are proved in Theorem 2.26 in next Section.

2.2 *A priori* bounds, existence and uniqueness of weak solutions for $\varepsilon = 0$

In this section, we prove some *a priori* estimates for $\varepsilon = 0$ and for $\varepsilon > 0$ that will be used both in proving existence of a weak solution of (1.4) when $\varepsilon = 0$ and in the next section, to study the limit of (1.4) as $k \to \infty$.

The next bound for ku^kv^k is key in the following. The proof is similar to Lemma 2.10, but here the estimates are proved over S_T which is unbounded. The strategy to obtain the estimate is to consider the integral over $(1, \infty) \times (0, T)$ by studying the equation of u^k and the integral over $(0, 1) \times (0, T)$ by studying the equation of v^k.

Lemma 2.18. *There exists a constant $C > 0$, independent of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.4), we have*

$$\iint_{S_T} k u^k v^k \mathrm{d}x\mathrm{d}t \leq C.$$

Proof. Multiplying the equation for u^k by the cut-off function φ^L defined in Lemma 2.10 and integrating over $\mathbb{R}^+ \times (0, t_0)$ where $t_0 \in (0, T]$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+} \varphi^L u^k(x, t_0)\mathrm{d}x + k\int_0^{t_0}\int_{\mathbb{R}^+} \varphi^L u^k v^k \mathrm{d}x\mathrm{d}t \\
&= \int_0^{t_0}\int_{\mathbb{R}^+} \varphi^L_{xx}\phi(u^k)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} \varphi^L u^k_0 \mathrm{d}x,
\end{aligned}$$

it follows that there exists some $C > 0$ such that

$$\int_1^L \varphi^L u^k(x, t_0)\mathrm{d}x + k\int_0^{t_0}\int_1^L \varphi^L u^k v^k \mathrm{d}x\mathrm{d}t \leq C, \tag{2.29}$$

independently of L, $k > 0$, by (2.28), the definition of φ^L and the fact $u^k_0 \in L^1(\mathbb{R}^+)$. Letting $L \to \infty$ and using Lebesgue's Monotone Convergence Theorem give

$$k\int_0^T\int_1^\infty u^k v^k \mathrm{d}x\mathrm{d}t \leq C. \tag{2.30}$$

Similarly, multiplying the equation for v^k by $\hat{\varphi}$ where $\hat{\varphi} \in C^\infty(\mathbb{R}^+)$ is such that $0 \leq \hat{\varphi}(x) \leq 1$, $\hat{\varphi}(x) = 1$ for $x \in [0, 1]$ and $\hat{\varphi}(x) = 0$ for all $x \geq 2$, then integrating over $\mathbb{R}^+ \times (0, t_0)$ for $t_0 \in (0, T]$ yields

$$k\int_0^{t_0}\int_{\mathbb{R}^+} \hat{\varphi} u^k v^k \mathrm{d}x\mathrm{d}t = \varepsilon\int_0^{t_0}\int_{\mathbb{R}^+} \hat{\varphi}_{xx}\phi(v^k)\mathrm{d}x\mathrm{d}t - \int_0^2 \hat{\varphi}\left[v^k(x, t_0) - v^k_0\right]\mathrm{d}x,$$

together with (2.28), imply that

$$k\int_0^T\int_0^1 u^k v^k \mathrm{d}x\mathrm{d}t \leq C, \tag{2.31}$$

independently of $k > 0$ and of $\varepsilon \geq 0$. Then the result follows from (2.29) and (2.31). $\square$

In the following, we prove that u^k and $|v^k(x,t_0) - V_0|$ are bounded independently of ε in $L^1(\mathbb{R}^+)$.

Lemma 2.19. *There exists a constant $C > 0$, independent of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.4), we have*

$$\int_{\mathbb{R}^+} u^k(x,t_0)\mathrm{d}x \leq C \quad \text{and} \quad \int_{\mathbb{R}^+} |v^k(x,t_0) - V_0|\mathrm{d}x \leq C, \tag{2.32}$$

for all $t_0 \in [0,T]$.

Proof. The estimate for u^k is immediately from (2.29) and the Monotone Convergence Theorem.

Now choose a smooth convex function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \geq 0,\ m(0) = 0,\ m'(0) = 0,\ m(r) = |r| - \frac{1}{2} \text{ for } |r| > 1.$$

For each $\alpha > 0$, define the functions

$$m_\alpha(r) := \alpha m(\frac{r}{\alpha}),$$

which approximate the modulus function as $\alpha \to 0$. Denote $\hat{v} = v^k - V_0$, $\hat{z} = \phi(v^k) - \phi(V_0)$.

Now with ψ^L as in the proof of Lemma 2.11, we have

$$\begin{aligned}\int_{\mathbb{R}^+} m'_\alpha(\hat{z})\psi^L \hat{z}_{xx}\mathrm{d}x &= -\int_{\mathbb{R}^+} m'_\alpha(\hat{z})\psi^L_x \hat{z}_x\mathrm{d}x - \int_{\mathbb{R}^+} m''_\alpha(\hat{z})\psi^L|\hat{z}_x|^2\mathrm{d}x \\ &\leq -\int_{\mathbb{R}^+} \left[m_\alpha(\hat{z})\right]_x \psi^L_x\mathrm{d}x \\ &= \int_{\mathbb{R}^+} m_\alpha(\hat{z})\psi^L_{xx}\mathrm{d}x.\end{aligned}$$

Multiplying the equation of $\hat{v}$ by $m'_\alpha(\hat{z})\psi^L$ and integrating over $\mathbb{R}^+ \times (0, t_0)$, we obtain

$$\begin{aligned}\int_0^{t_0}\int_{\mathbb{R}^+} m'_\alpha(\hat{z})\psi^L \hat{v}_t \mathrm{d}x\mathrm{d}t =& \varepsilon\int_0^{t_0}\int_{\mathbb{R}^+} m'_\alpha(\hat{z})\psi^L \hat{z}_{xx}\mathrm{d}x\mathrm{d}t \\ &-k\int_0^{t_0}\int_{\mathbb{R}^+} m'_\alpha(\hat{z})\psi^L u^k v^k \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Letting $\alpha \to 0$ and [14, Lemma 7.6] yields

$$\begin{aligned}&\int_{\mathbb{R}^+} |\hat{v}(x,t_0)|\psi^L \mathrm{d}x - \int_{\mathbb{R}^+} |\hat{v}(x,0)|\psi^L \mathrm{d}x \\ \leq& \varepsilon\int_0^{t_0}\int_{\mathbb{R}^+} |\hat{z}|\psi^L_{xx}\mathrm{d}x\mathrm{d}t - k\int_0^{t_0}\int_{\mathbb{R}^+} \mathrm{sgn}(\hat{z})\psi^L u^k v^k \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.33}$$

We know that $\psi^L_{xx} \neq 0$ only when $\psi^L_{xx} \in [L, L+1]$ and by Lemma 2.18, the right-hand side of (2.33) is bounded independently of L and k. So it follows from the fact that $v_0^k - v_0^\infty$ is bounded in $L^1(\mathbb{R}^+)$, there exists $C > 0$ independent of k, such that for all $t_0 \in [0, T]$

$$\int_{\mathbb{R}^+} |v^k(x, t_0) - V_0|\mathrm{d}x \leq C.$$

□

By using the Mean Value Theorem and *a priori* bounds on u^k, v^k on Corollary 2.17, we can get the following corollary of Lemma 2.19.

Corollary 2.20. *There exists a constant $C > 0$, independent of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.4), we have*

$$\int_{\mathbb{R}^+} \phi(u^k)(\cdot, t_0)\mathrm{d}x \leq C \quad \text{and} \quad \int_{\mathbb{R}^+} |\phi(v^k)(\cdot, t_0) - \phi(V_0)|\mathrm{d}x \leq C, \tag{2.34}$$

for all $t_0 \in [0, T]$.

Next we prove a bound for the L^2-norm of the space derivatives $\phi(u^k)_x$ and $\phi(v^k)_x$.

Lemma 2.21. *There exists $C > 0$, independent of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.4),*

$$\iint_{S_T} |\phi(u^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C, \quad \text{and} \quad \varepsilon \iint_{S_T} |\phi(v^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C. \tag{2.35}$$

Proof. The proof follows the similar arguments in Lemma 2.11. Let ψ^L be as in the proof of Lemma 2.11. Then multiplying the equation for u^k by $\big[\phi(u^k) - \phi(U_0)\big]\psi^L$ and integrating over S_T give

$$\begin{aligned}\iint_{S_T} \big[\phi(u^k) - \phi(U_0)\big]\psi^L u^k_t \mathrm{d}x\mathrm{d}t = &\int_{\mathbb{R}^+} \big[F(x,T) - F(x,0)\big]\psi^L \mathrm{d}x \\ &+ \int_{\mathbb{R}^+} \phi(U_0)\big(u^k_0 - u^k(x,T)\big)\psi^L \mathrm{d}x,\end{aligned}$$

where $F = \displaystyle\int_0^{u^k} \phi(s)\mathrm{d}s.$

Thus

$$\begin{aligned}\iint_{S_T} |\phi(u^k)_x|^2 \psi^L \mathrm{d}x\mathrm{d}t = &- \int_{\mathbb{R}^+} \phi(U_0)\big(u^k_0 - u^k(x,T)\big)\psi^L \mathrm{d}x \\ &- \int_{\mathbb{R}^+} \big[F(x,T) - F(x,0)\big]\psi^L \mathrm{d}x \\ &+ \frac{1}{2}\iint_{S_T} \big[\phi(u^k) - \phi(U_0)\big]^2 \psi^L_{xx} \mathrm{d}x\mathrm{d}t \\ &- k\iint_{S_T} \big[\phi(u^k) - \phi(U_0)\big]\psi^L u^k v^k \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Here the second term on the right-hand side is an integral over the unbounded domain $\mathbb{R}^+$. By the Mean Value Theorem and (2.28), we obtain for $s \in [0, U_0]$ such that

$$F(x,T) - F(x,0) = \phi(s)u^k(x,T),$$

which yields

$$\left|\int_{\mathbb{R}^+} [F(x,T) - F(x,0)]\psi^L \mathrm{d}x\right| \leq \phi(U_0) \sup_{0\leq t\leq T} \int_{\mathbb{R}^+} u^k(x,t)\mathrm{d}x,$$

which is bounded by Lemma 2.19. Combining with Lemma 2.18, using Lebesgue's Monotone Convergence Theorem and letting $L \to \infty$ imply that there exists a constant $C > 0$ such that

$$\iint_{S_T} |\phi(u^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C,$$

independently of k. If $\varepsilon > 0$, the estimate for $\phi(v^k)_x$ can be proved likewise, using the equation for v^k. $\square$

The following estimates for the differences of space and time translates of solutions will yield sufficient compactness both to obtain the existence of solutions of (1.4) when $\varepsilon > 0$ and $\varepsilon = 0$, and to study the strong-interaction limit $k \to \infty$. The estimates for the differences of space translates of solutions are proved in the similar way to Lemma 2.15 [10], which importantly allows $\varepsilon = 0$. Note that we need alternative procedures to deal with the nonlinear diffusion, and the monotonicity properties of ϕ and Lemma 7.6 [14] are used here.

Recall the notion for space and time translates introduced in (2.23).

Lemma 2.22. *Suppose that $\varepsilon \geq 0$ and let (u^k, v^k) be a solution of (1.4) satisfying (2.28). Then for each $r \in (0,1)$, there exists a function $K_r \geq 0$ independent of $\varepsilon \geq 0$ and $k > 0$ such that $K_r(\delta) \to 0$ as $|\delta| \to 0$ and for all $|\delta| \leq \frac{r}{4}$ and $t \in (0,T)$, we have*

$$\int_r^\infty \left|\phi(u^k) - \phi(S_\delta u^k)\right| + \left|\phi(v^k) - \phi(S_\delta v^k)\right| \mathrm{d}x \leq K_r(\delta).$$

Proof. Let

$$u := u^k - S_\delta u^k, \quad w := \phi(u^k) - \phi(S_\delta u^k),$$

$$v := v^k - S_\delta v^k, \quad z := \phi(v^k) - \phi(S_\delta v^k), \tag{2.36}$$

and define a cut-off function $\gamma_r^1 \in C^\infty(\mathbb{R}^+)$ such that $0 \le \gamma_r^1 \le 1$ and

$$\gamma_r^1(x) = \begin{cases} 0 & x \in [0, r/2], \\ 1 & x \in [r, 1], \\ 0 & x \ge 2. \end{cases}$$

Then given $L \ge 1$ define a family of cut-off function $\gamma_r^L \in C^\infty(\mathbb{R}^+)$

$$\gamma_r^L(x) = \begin{cases} \gamma_r^1(x) & x \in [0, r], \\ 1 & x \in [r, L], \\ \gamma_r^1(x + 1 - L) & x \ge L. \end{cases}$$

Note that $0 \le \gamma_r^L \le 1$ for all L, and $(\gamma_r^L)_x$, $(\gamma_r^L)_{xx}$ are bounded in both $L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)$ independently of L. Then

$$\begin{aligned} &u_t = w_{xx} - k\left(u^k v^k - S_\delta u^k S_\delta v^k\right), && \text{in } \left(\frac{r}{4}, \infty\right) \times (0, T), \\ &v_t = \varepsilon z_{xx} - k\left(u^k v^k - S_\delta u^k S_\delta v^k\right), && \text{in } \left(\frac{r}{4}, \infty\right) \times (0, T), \\ &u(x, 0) = u_0^k - S_\delta u_0^k, \quad v(x, 0) = v_0^k - S_\delta v_0^k, && \text{for } x \in \left(\frac{r}{4}, \infty\right). \end{aligned}$$

Let m_α be as defined in the proof of Lemma 2.19, multiplying the equation for u by $m'_\alpha(w)\gamma_r^L$ and integrating over $\left(\frac{r}{2}, \infty\right) \times (0, t_0)$ give

$$\begin{aligned} &\int_0^{t_0} \int_{\mathbb{R}^+} m'_\alpha(w) \gamma_r^L u_t \mathrm{d}x\mathrm{d}t \\ = &- \int_0^{t_0} \int_{\mathbb{R}^+} m''_\alpha(w) |w_x|^2 \gamma_r^L \mathrm{d}x\mathrm{d}t + \int_0^{t_0} \int_{\mathbb{R}^+} m_\alpha(w) \left(\gamma_r^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ &- k \int_0^{t_0} \int_{\mathbb{R}^+} m'_\alpha(w) \gamma_r^L \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t \\ \le &\int_0^{t_0} \int_{\mathbb{R}^+} m_\alpha(w) \left(\gamma_r^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ &- k \int_0^{t_0} \int_{\mathbb{R}^+} m'_\alpha(w) \gamma_r^L \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t, \end{aligned}$$

letting $\alpha \to 0$ and by [14, Lemma 7.6], we have

$$\int_{\frac{r}{2}}^{\infty} |u(x,t_0)|\gamma_r^L \mathrm{d}x \le \int_{\frac{r}{2}}^{\infty} |u(x,0)|\gamma_r^L \mathrm{d}x + \int_0^{t_0}\int_{\frac{r}{2}}^{\infty} |w| \left(\gamma_r^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ - k\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \gamma_r^L \mathrm{sgn}(w)\left(u^k v^k - S_\delta u^k S_\delta v^k\right)\mathrm{d}x\mathrm{d}t, \quad (2.37)$$

similarly

$$\int_{\frac{r}{2}}^{\infty} |v(x,t_0)|\gamma_r^L \mathrm{d}x \le \int_{\frac{r}{2}}^{\infty} |v(x,0)|\gamma_r^L \mathrm{d}x + \varepsilon\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} |z| \left(\gamma_r^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ - k\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \gamma_r^L \mathrm{sgn}(z)\left(u^k v^k - S_\delta u^k S_\delta v^k\right)\mathrm{d}x\mathrm{d}t. \quad (2.38)$$

Adding (2.37) and (2.38) then gives

$$\begin{aligned} &\int_{\frac{r}{2}}^{\infty} \gamma_r^L \left\{|u(x,t_0)| + |v(x,t_0)|\right\}\mathrm{d}x \\ \le &\int_{\frac{r}{2}}^{\infty} \gamma_r^L \left\{|u(x,0)| + |v(x,0)|\right\}\mathrm{d}x + \int_0^{t_0}\int_{\frac{r}{2}} \left(\gamma_r^L\right)_{xx}\left\{|w| + \varepsilon|z|\right\}\mathrm{d}x\mathrm{d}t \\ &- k\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \gamma_r^L\left[\mathrm{sgn}(w) + \mathrm{sgn}(z)\right]\left(u^k v^k - S_\delta u^k S_\delta v^k\right)\mathrm{d}x\mathrm{d}t \\ \le &\int_{\frac{r}{2}}^{\infty} \gamma_r^L \left\{|u(x,0)| + |v(x,0)|\right\}\mathrm{d}x + \int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx}\left\{|w| + \varepsilon|z|\right\}\mathrm{d}x\mathrm{d}t, \end{aligned} \quad (2.39)$$

because

$$\left[\mathrm{sgn}(w) + \mathrm{sgn}(z)\right]\left(u^k v^k - S_\delta u^k S_\delta v^k\right) \ge 0. \quad (2.40)$$

Now we prove the following bound for right-hand side of (2.39),

$$\begin{aligned} &\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx}|w|\mathrm{d}x\mathrm{d}t \\ =&\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx}|\phi(u^k)(x,t) - \phi(u^k)(x+\delta,t)|\mathrm{d}x\mathrm{d}t \\ =&\int_0^{t_0}\int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx}\left|\int_0^1 \delta\phi(u^k)_x(x+\theta\delta,t)\mathrm{d}\theta\mathrm{d}x\mathrm{d}t\right| \end{aligned}$$

$$\leq |\delta| \int_0^{t_0} \int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx} \int_0^1 \left|\phi(u^k)_x(x+\theta\delta,t)\right| \mathrm{d}\theta \mathrm{d}x \mathrm{d}t$$
$$\leq |\delta| \left[\int_0^{t_0} \int_{\frac{r}{4}}^{\infty} \left(\gamma_r^L\right)_{xx}^2 \mathrm{d}x \mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^{t_0} \int_{\frac{r}{4}}^{\infty} \left|\phi(u^k)_x(x+\delta,t)\right|^2 \mathrm{d}x \mathrm{d}t\right]^{\frac{1}{2}}.$$

By Lemma 2.21 and a similar estimate for z, we get

$$\int_0^{t_0} \int_{\frac{r}{2}}^{\infty} \left(\gamma_r^L\right)_{xx} \{|w| + \varepsilon |z|\} \mathrm{d}x \mathrm{d}t \leq K_r |\delta|, \tag{2.41}$$

for some constant K_r. The result follows from (2.39), the fact that $\|u_0^k(\cdot + \delta) - u_0^k(\cdot)\|_{L^1((r,\infty))} + \|v_0^k(\cdot + \delta) - v_0^k(\cdot)\|_{L^1((r,\infty))} \leq \omega_r(\delta)$ where $\omega_r(\mu) \to 0$ as $\mu \to 0$ and Lebesgue's Monotone Convergence Theorem combining with the Mean Value Theorem. □

The estimates for the difference of time translates are proved by using similar methods to those in the proof of Lemma 2.13, passing to the limit as $L \to \infty$ in integrals over $(0, L+1)$ to obtain estimates on integrals over $\mathbb{R}^+$.

Lemma 2.23. *Suppose $\varepsilon \geq 0$ and let (u^k, v^k) be a solution of (1.4) satisfying (2.28). Then there exists $C > 0$, independent of ε and k, for any $\tau \in (0, T)$ that*

$$\int_0^{T-\tau} \int_{\mathbb{R}^+} |\phi(T_\tau u^k) - \phi(u^k)|^2 \mathrm{d}x \mathrm{d}t \leq \tau C,$$
$$\int_0^{T-\tau} \int_{\mathbb{R}^+} |\phi(T_\tau v^k) - \phi(v^k)|^2 \mathrm{d}x \mathrm{d}t \leq \tau C.$$

Proof. Let ψ^L be as in the proof of Lemma 2.11. Then it follows using the Mean Value Theorem such that

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L|\phi(T_\tau u^k)-\phi(u^k)|^2\mathrm{d}x\mathrm{d}t\\
\leq &N\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L\left[\phi(T_\tau u^k)-\phi(u^k)\right]\left(T_\tau u^k-u^k\right)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $N=\phi'(U_0)$, because ϕ is increasing, and ϕ' is bounded by N for $s\in[0,U_0]$, then we have

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L|\phi(T_\tau u^k)-\phi(u^k)|^2\mathrm{d}x\mathrm{d}t\\
\leq &N\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L\left[\phi\left(T_\tau u^k\right)-\phi(u^k)\right]\phi(u^k)_{xx}(x,t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s\\
&-Nk\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L\left[\phi(T_\tau u^k)-\phi(u^k)\right]T_\tau u^kT_\tau v^k\mathrm{d}x\mathrm{d}t\mathrm{d}s\\
= &I_1+I_2+I_3,
\end{aligned}$$

with

$$\begin{aligned}
I_1 &:= -N\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L\left[\phi(T_\tau u^k)-\phi(u^k)\right]_x\phi(u^k)_x(x,t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s,\\
I_2 &:= -N\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L_x\left[\phi(T_\tau u^k)-\phi(u^k)\right]\phi(u^k)(x,t+s)\mathrm{d}x\mathrm{d}t\mathrm{d}s,\\
I_3 &:= -Nk\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}^+}\psi^L\left[\phi(T_\tau u^k)-\phi(u^k)\right]T_\tau u^kT_\tau v^k\mathrm{d}x\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

I_1 can be split into two terms and by the Cauchy-Schwarz inequality, using the property $\psi^L\leq 1$ and letting $L\to\infty$ yield

$$|I_1|\leq 2\tau N\left\{\int_0^T\int_{\mathbb{R}^+}\left|\phi(u^k)_x\right|^2\mathrm{d}x\mathrm{d}t\right\}.$$

which is bounded by Lemma 2.21.

Then using the bounded for $\sup\left|\psi^L_x\right|$ independently of L, we have

$$|I_2|\leq\left(\sup\left|\psi^L_x\right|\right)N\int_0^{\tau}\int_0^{T-\tau}\int_L^{L+1}\left|\phi(T_\tau u^k)-\phi(u^k)\right|\left|\phi(u^k)_x(x,t+s)\right|\mathrm{d}x\mathrm{d}t\mathrm{d}s,$$

which vanishes as $L \to \infty$, since (2.28), Lemma 2.19 and Lemma 2.21.

The last term is easier to handle, by (2.28) and letting $L \to \infty$ we get

$$|I_3| \le 2M\tau N \int_0^T \int_{\mathbb{R}^+} u^k v^k \mathrm{d}x\mathrm{d}t,$$

which is bounded by Lemma 2.18. When $\varepsilon > 0$, the estimate for v^k follows likewise, using the equation for v^k. When $\varepsilon = 0$, a similar but simpler argument applies, omitting the terms involving $\phi(v^k)_{xx}$. $\square$

The following lemma will be used to prove the convergence results of (u^k, v^k) as $\varepsilon \to 0$ and $k \to \infty$ and to study the weak solutions of $k \to \infty$ limit problems.

Lemma 2.24. *Let (u^k, v^k) be weak solutions of (1.4) with $k > 0$ and $\varepsilon \ge 0$. Then*

$$\phi(u^k) - \phi(\hat{u}) \in L^2(0, T; W_0^{1,2}(\mathbb{R}^+)), \tag{2.42}$$

and

$$\varepsilon[\phi(v^k) - \phi(V_0)] \in L^2(0, T; W^{1,2}(\mathbb{R}^+)), \tag{2.43}$$

where $\hat{u} \in C^\infty(\mathbb{R}^+)$ is a smooth function such that $\hat{u} = U_0$ when $x = 0$ and $\hat{u} = 0$ when $x > 1$.

Proof. The result for u^k follows from Corollary 2.17 and Corollary 2.20 which ensure that $\phi(u^k) - \phi(U_0) \in L^\infty(S_T)$ and $\phi(u^k) - \phi(U_0) \in L^1(S_T)$, together with Lemma 2.21 which ensures that $\phi(u^k)_x \in L^2(S_T)$. If $\varepsilon > 0$, the estimates for v^k can be proved likewise. $\square$

We can now prove a convergence result for solutions (u^k, v^k) of (1.4) as $\varepsilon \to 0$.

Lemma 2.25. *Let $k > 0$ be fixed and $(u_\varepsilon^k, v_\varepsilon^k)$ be solution of (1.4) satisfying (2.28) with $\varepsilon > 0$. Then there exist $(u_\star^k, v_\star^k) \in (L^\infty(S_T))^2$ such that up to a subsequence, for each $J > 0$*

$$\begin{aligned}
\phi(u_\varepsilon^k) &\to \phi(u_\star^k) && \text{in} \quad L^2((0,J)\times(0,T)),\\
u_\varepsilon^k &\to u_\star^k && \text{a.e. in} \quad (0,J)\times(0,T),\\
\phi(v_\varepsilon^k) &\to \phi(v_\star^k) && \text{in} \quad L^2((0,J)\times(0,T)),\\
v_\varepsilon^k &\to v_\star^k && \text{a.e. in} \quad (0,J)\times(0,T),\\
\phi(u_\varepsilon^k) - \phi(\hat{u}) &\rightharpoonup \phi(u_\star^k) - \phi(\hat{u}) && \text{in} \quad L^2\left(0,T; W_0^{1,2}(\mathbb{R}^+)\right),
\end{aligned}$$

as $\varepsilon \to 0$, where $\hat{u} \in C^\infty(\mathbb{R}^+)$ is a smooth function that $\hat{u} = U_0$ when $x = 0$ and $\hat{u} = 0$ when $x > 1$.

Proof. It follows from Lemma 2.19, Lemma 2.21 and Corollary 2.17 that $\phi(u_\varepsilon^k)$ and $\phi(v_\varepsilon^k) - \phi(V_0)$ are bounded independently of $\varepsilon \geq 0$ in $L^2(S_T)$. By Lemma 2.22 and Lemma 2.23, using the Riesz-Fréchet-Kolmogorov Theorem [3, Theorem 4.26], yield that the sets $\left\{\phi(v_\varepsilon^k) - \phi(V_0)\right\}_{\varepsilon>0}$ and $\left\{\phi(u_\varepsilon^k)\right\}_{\varepsilon>0}$ are each relatively compact in $L^2\left((0,J)\times(0,T)\right)$ for each $J > 0$. The weak convergence of $\phi(u_\varepsilon^k) - \phi(\hat{u})$ in $L^2\left(0,T; W_0^{1,2}(\mathbb{R}^+)\right)$ follows from Lemma 2.24. Then we know that $\phi(u_\varepsilon^k) \to \phi(u_\star^k)$ and $\phi(v_\varepsilon^k) \to \phi(v_\star^k)$ almost everywhere in $(0,J)\times(0,T)$, so since ϕ^{-1} is continuous, then we have $u_\varepsilon^k \to \phi^{-1}(\phi(u_\star^k))$ and $v_\varepsilon^k \to \phi^{-1}(\phi(v_\star^k))$ almost everywhere in $(0,J)\times(0,T)$. $\square$

Recall that

$$\mathcal{F}_T := \left\{\xi \in C^1(S_T) : \ \xi(0,t) = \xi(\cdot,T) = 0 \text{ for } t \in (0,T) \text{ and } \operatorname{supp}\xi \subset [0,J]\times[0,T] \right. \\ \left. \text{for some } J > 0\right\}.$$

and

$$\Omega_J := \left\{\alpha \in W^{1,2}(0,J) | \ \alpha = 0 \text{ at } x = 0\right\}.$$

Lemma 2.15 and Lemma 2.25 enable the following result to be established.

Theorem 2.26. *Let $\varepsilon = 0$ and $k > 0$. Then Problem (1.4) has a unique weak solution $(u^k, v^k) \in (L^\infty(S_T))^2$ for each $J > 0$ such that*

(i) *$\phi(u^k) \in \phi(\hat{u}) + L^2(0, T; \Omega_J)$, where $\hat{u} \in C^\infty(\mathbb{R}^+)$ is a smooth function that $\hat{u} = U_0$ when $x = 0$ and $\hat{u} = 0$ when $x > 1$;*

(ii) *(u^k, v^k) satisfies*

$$\int_{\mathbb{R}^+} u_0^k \xi(x, 0)\mathrm{d}x + \iint_{S_T} \xi_t u^k \mathrm{d}x\mathrm{d}t = \iint_{S_T} \xi_x \phi(u^k)_x \mathrm{d}x\mathrm{d}t + k \iint_{S_T} \xi u^k v^k \mathrm{d}x\mathrm{d}t, \tag{2.44}$$

$$\int_{\mathbb{R}^+} v_0^k \xi(x, 0)\mathrm{d}t + \iint_{S_T} \xi_t v^k \mathrm{d}x\mathrm{d}t = k \iint_{S_T} \xi u^k v^k \mathrm{d}x\mathrm{d}t, \tag{2.45}$$

for all $\xi \in \mathcal{F}_T$.

Proof. Multiplying (1.4) by $\xi \in \mathcal{F}_T$ and integrating over S_T yield that for each $\varepsilon > 0$, solution $(u_\varepsilon^k, v_\varepsilon^k)$ of (1.4) satisfy

$$\int_{\mathbb{R}^+} u_0^k \xi(x, 0)\mathrm{d}x + \iint_{S_T} \xi_t u_\varepsilon^k \mathrm{d}x\mathrm{d}t = \iint_{S_T} \xi_x \phi(u_\varepsilon^k)_x \mathrm{d}x\mathrm{d}t + k \iint_{S_T} \xi u_\varepsilon^k v_\varepsilon^k \mathrm{d}x\mathrm{d}t, \tag{2.46}$$

$$\int_{\mathbb{R}^+} v_0^k \xi(x, 0)\mathrm{d}t + \iint_{S_T} \xi_t v_\varepsilon^k \mathrm{d}x\mathrm{d}t = \varepsilon \iint_{S_T} \xi_x \phi(v_\varepsilon^k)_x \mathrm{d}x\mathrm{d}t + k \iint_{S_T} \xi u_\varepsilon^k v_\varepsilon^k \mathrm{d}x\mathrm{d}t. \tag{2.47}$$

Then the existence of a solution (u^k, v^k) to (2.44)-(2.45) follows by using Lemma 2.25 to pass to the limit along a subsequence as $\varepsilon \to 0$ in (2.46)-(2.47). The uniqueness of (u^k, v^k) follows from the comparison principle proved in Lemma 2.15. $\square$

2.3 Limit problem for (1.4) as $k \to \infty$

The *a priori* estimates of the previous section yield sufficient compactness to establish the existence of limits of solutions of (1.4) as $k \to \infty$, both when $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 2.27. *Let $\varepsilon \geq 0$ be fixed and (u^k, v^k) be weak solutions of (1.4) satisfying (2.28) with $k > 0$. Then there exists $(u, v) \in (L^\infty(S_T))^2$ such that up to a subsequence, for each $J > 0$ that*

$$\begin{aligned}
\phi(u^k) &\to \phi(u) && \text{in} \quad L^2((0,J)\times(0,T)),\\
u^k &\to u && \text{a.e. in} \quad (0,J)\times(0,T),\\
\phi(v^k) &\to \phi(v) && \text{in} \quad L^2((0,J)\times(0,T)),\\
v^k &\to v && \text{a.e. in} \quad (0,J)\times(0,T),\\
\phi(u^k) - \phi(\hat{u}) &\rightharpoonup \phi(u) - \phi(\hat{u}) && \text{in} \quad L^2\left(0,T; W_0^{1,2}(\mathbb{R}^+)\right),
\end{aligned}$$

and for $\varepsilon > 0$

$$\phi(v^k) - \phi(V_0) \rightharpoonup \phi(v) - \phi(V_0) \qquad \text{in} \quad L^2\left(0,T; W^{1,2}(\mathbb{R}^+)\right),$$

as $k \to \infty$, where $\hat{u} \in C^\infty(\mathbb{R}^+)$ is a smooth function that $\hat{u} = U_0$ when $x = 0$ and $\hat{u} = 0$ when $x > 1$.

Proof. The proof is directly analogous to that of Lemma 2.25, using bounds independent of k in place of bounds independent of ε. The weak convergence of $\phi(v^k) - \phi(V_0)$ in $L^2(0,T; W^{1,2}(\mathbb{R}^+))$ follows from Lemma 2.24. $\square$

The following segregation result is a key to characterisation of the limits u, v in Lemma 2.27.

Lemma 2.28. *Let $\varepsilon \geq 0$ and (u,v) be as in Lemma 2.27. Then*

$$uv = 0 \text{ a.e. in } S_T. \tag{2.48}$$

Proof. It follows from Lemma 2.18 and Lemma 2.27 that $uv = 0$ almost everywhere in S_T combining with Lemma 2.17 and using Lebesgue's Dominated Convergence Theorem. $\square$

To derive the limit problem, we set

$$w^k := u^k - v^k, \quad w := u - v. \tag{2.49}$$

Then it follows from Lemma 2.27 and Lemma 2.28 that as a sequence $k_n \to \infty$,

$$w^{k_n} \to w \quad \text{in } L^2\left((0, J) \times (0, T)\right) \text{ for all } J > 0 \text{ and a.e in } S_T,$$

and that

$$u = w^+, \quad v = -w^-,$$

where $s^+ = \max\{0, s\}$ and $s^- = \min\{0, s\}$.

Lemma 2.29. *Let $\varepsilon \geq 0$ and (u, v) be as in Lemma 2.27. Then*

$$\begin{aligned} &\iint_{S_T} (u - v)\xi_t \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} (u_0^\infty - v_0^\infty)\xi(x, 0)\mathrm{d}x \\ = &\iint_{S_T} (\phi(u) - \varepsilon\phi(v))_x \xi_x \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.50}$$

for all

$$\xi \in \mathcal{F}_T := \left\{\xi \in C^1(S_T) : \ \xi(0, t) = \xi(\cdot, T) = 0 \text{ for } t \in (0, T) \text{ and } \operatorname{supp} \xi \subset [0, J] \times [0, T] \right.$$
$$\left. \text{for some } J > 0\right\}.$$

Proof. Multiplying the difference between the equations for u^k and v^k by $\xi \in \mathcal{F}_T$ and integrating over S_T gives

$$\begin{aligned}&\iint_{S_T} (u^k - v^k)\xi_t \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} (u_0^k - v_0^k)\xi(x,0)\mathrm{d}x \\ =& \iint_{S_T} (\phi(u^k) - \varepsilon\phi(v^k))_x \xi_x \mathrm{d}x\mathrm{d}t,\end{aligned}$$

for which (2.50) follows using Lemma 2.27 and the fact that $u_0^k \to u_0^\infty$ and $v_0^k \to v_0^\infty$ in $L^1(\mathbb{R}^+)$ as $k \to \infty$. $\square$

Now define

$$\mathcal{D}(s) := \begin{cases} \phi(s) & s \geq 0, \\ -\varepsilon\phi(-s) & s < 0, \end{cases} \tag{2.51}$$

and the limit problem

$$\begin{cases} w_t = \mathcal{D}(w)_{xx}, & \text{in } S_T, \\ w(x,0) = w_0(x) := -V_0, & \text{for } x > 0, \\ w(0,t) = U_0, & \text{for } t \in (0,T). \end{cases} \tag{2.52}$$

Definition 2.30. *A function w is a weak solution of (2.52) if*

(i) $w \in L^\infty(S_T)$,

(ii) $\mathcal{D}(w) \in \mathcal{D}(\hat{w}) + L^2(0,T; W_0^{1,2}(\mathbb{R}^+))$, *where $\hat{w} \in C^\infty(\mathbb{R}^+)$ is a smooth function with $\hat{w} = U_0$ when $x = 0$ and $\hat{w} = -V_0$ when $x > 1$,*

(iii) *w satisfies*

$$\int_{\mathbb{R}^+} w_0(x)\xi(x,0)\mathrm{d}x + \iint_{S_T} w\xi_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} \mathcal{D}(w)_x \xi_x \mathrm{d}x\mathrm{d}t. \tag{2.53}$$

for all $\xi \in \mathcal{F}_T$.

Theorem 2.31. *Let $\varepsilon \geq 0$. The function w defined in (2.49) is a weak solution of problem (2.52) and the whole sequence (u^k, v^k) in Lemma 2.27 converges to $(w^+, -w^-)$.*

Proof. The existence of a weak solution is a straight forward consequence of Definition 2.30 and Lemma 2.29. The fact that the whole sequence (u^k, v^k) converges to $(w^+, -w^-)$ follows from the uniqueness results proved in Theorem 2.32 when $\varepsilon > 0$ and in Theorem 2.33 when $\varepsilon = 0$. $\square$

Now we prove the uniqueness of the weak solution of (2.52) for $\varepsilon > 0$.

Theorem 2.32. *Let $\varepsilon > 0$, then there exists at most one solution w of the limit problem (2.52).*

Proof. Let w_1, w_2 be two weak solution of problem (2.52) and let $\xi \in W^{1,2}((0,J) \times (0,T))$ be defined by

$$\xi(x,t) = \int_t^T [\mathcal{D}(w_1)(x,\tau) - \mathcal{D}(w_2)(x,\tau)]\psi^L \mathrm{d}\tau. \tag{2.54}$$

where ψ^L as in proof of Lemma 2.11.

We know there exists $\xi_m \in \mathcal{F}_T$ such that $\xi_m \to \xi$ in $W^{1,2}((0,L) \times (0,T))$ by the Meyers-Serrin Theorem (see [22] or [1, p.66]).

Now, subtracting (2.53) for w_1 and w_2 yields

$$\iint_{S_T} (w_1 - w_2)\xi_{mt} \mathrm{d}x\mathrm{d}t = \iint_{S_T} (\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x)\xi_{mx} \mathrm{d}x\mathrm{d}t,$$

with $\xi_m \in \mathcal{F}_T$. Then letting $m \to \infty$, we deduce that for $\xi \in W^{1,2}((0,L) \times (0,T))$ defined in (2.54) satisfies

$$\iint_{S_T} (w_1 - w_2)\xi_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} (\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x)\xi_x \mathrm{d}x\mathrm{d}t.$$

Then we have

$$
\begin{aligned}
0 =& \iint_{S_T} (w_1 - w_2)(\mathcal{D}(w_1) - \mathcal{D}(w_2))\psi^L \mathrm{d}x\mathrm{d}t \\
&- \iint_{S_T} [\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x] \int_t^T [\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x]\psi^L \mathrm{d}\tau\mathrm{d}x\mathrm{d}t \\
&+ \iint_{S_T} [\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x] \int_t^T [\mathcal{D}(w_1) - \mathcal{D}(w_2)]\psi^L_x \mathrm{d}\tau\mathrm{d}x\mathrm{d}t \\
=& I_1 + I_2 + I_3
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 \colon &= \iint_{S_T} (w_1 - w_2)[\mathcal{D}(w_1) - \mathcal{D}(w_2)]\psi^L \mathrm{d}x\mathrm{d}t \\
I_2 \colon &= \frac{1}{2}\int_{\mathbb{R}^+} \psi^L \left[\int_0^T (\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x)\mathrm{d}t\right]^2 \mathrm{d}x \\
I_3 \colon &= \iint_{S_T} \psi^L_x [\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x] \int_t^T [\mathcal{D}(w_1) - \mathcal{D}(w_2)]\mathrm{d}\tau\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Since ψ^L_x is bounded, then with a positive constant C we get

$$
\begin{aligned}
|I_3| \le& C\int_0^T \int_L^{L+1} |\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x| \int_t^T |\mathcal{D}(w_1) - \mathcal{D}(w_2)|\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \\
\le& C\left[\int_0^T \int_L^{L+1} |\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^T \int_L^{L+1} \left|\int_0^T \mathcal{D}(w_1) - \mathcal{D}(w_2)\mathrm{d}\tau\right|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}},
\end{aligned}
$$

by the Cauchy-Schwarz inequality, we obtain

$$
|I_3| \le CT\left[\int_0^T \int_L^{L+1} |\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^T \int_L^{L+1} |\mathcal{D}(w_1) - \mathcal{D}(w_2)|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}}.
$$

By using the Triangle inequality, we have

$$
\begin{aligned}
&\left[\int_0^T \int_L^{L+1} |\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \\
\le& \left[\left(\iint_{S_T} |\mathcal{D}(w_1)_x|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} + \left(\iint_{S_T} |\mathcal{D}(w_2)_x|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\right],
\end{aligned}
$$

which is bounded independently of L by Definition 2.30.

Therefore with K_T is a positive constant we have

$$|I_3| \leq K_T \left[\int_0^T \int_L^{L+1} |\mathcal{D}(w_1) - \mathcal{D}(w_2)|^2 \mathrm{d}x\mathrm{d}t \right]^{\frac{1}{2}}.$$

We know from Definition 2.30 that $\mathcal{D}(w_1) - \mathcal{D}(w_2) \in L^2(0, T; W_0^{1,2}(\mathbb{R}^+))$, then by Lebesgue's Dominated Convergence Theorem, we have $|I_3| \to 0$ as $L \to \infty$.

Therefore, since $\mathcal{D}$ is increasing, by Lebesgue's Monotone Convergence Theorem, we get

$$\begin{aligned} I_1 + I_2 \to & \iint_{S_T} (w_1 - w_2)(\mathcal{D}(w_1) - \mathcal{D}(w_2))\mathrm{d}x\mathrm{d}t \\ & + \frac{1}{2} \int_{\mathbb{R}^+} \left[\int_0^T (\mathcal{D}(w_1)_x - \mathcal{D}(w_2)_x)\mathrm{d}t \right]^2 \mathrm{d}x, \end{aligned}$$

as $L \to \infty$.

The result $w_1 = w_2$ follows from the fact that $I_1 + I_2$ is non-negative. □

The proof of Theorem 2.32 cannot refer to $\varepsilon = 0$, since $\mathcal{D}(w_1) - \mathcal{D}(w_2) = 0$ whenever w_1, w_2 are both negative. Therefore, we need an alternative method to prove the uniqueness of weak solution when $\varepsilon = 0$. The proof is inspired by [19, Proposition 5].

First, if we choose a smooth test function $\hat{\xi} \in C_0^\infty(\mathbb{R}^+ \times [0, T])$, the weak solution w satisfies

(i) $w \in L^\infty(S_T)$,

(ii) $\mathcal{D}(w) \in \mathcal{D}(\hat{w}) + L^2(0, T; W_0^{1,2}(\mathbb{R}^+))$, where $\hat{w} \in C^\infty(\mathbb{R}^+)$ is a smooth function with $\hat{w} = U_0$ when $x = 0$ and $\hat{w} = -V_0$ when $x > 1$,

(iii) w satisfies

$$\int_{\mathbb{R}^+} w_0(x)\hat{\xi}(x, 0)\mathrm{d}x + \iint_{S_T} w\hat{\xi}_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} \mathcal{D}(w)_x \hat{\xi}_x \mathrm{d}x\mathrm{d}t. \quad (2.55)$$

The proof of the following result also works for $\varepsilon > 0$, so we will state and prove it here for the general case $\varepsilon \geq 0$.

Theorem 2.33. *Let $\varepsilon \geq 0$ and consider two solutions $w, \tilde{w}$ of problem (2.52) with initial data $w_0, \tilde{w}_0$ respectively, then*

$$\iint_{S_T} |w - \tilde{w}| \mathrm{d}x\mathrm{d}t \leq C(T) \int_{\mathbb{R}^+} |w_0 - \tilde{w}_0| \mathrm{d}x, \tag{2.56}$$

and there exists at most one solution of problem (2.52) for given initial function w_0.

Proof. We know that there exists $\hat{\xi}_m \in C_0^\infty(\mathbb{R}^+ \times [0,T])$ such that $\hat{\xi}_m \to \hat{\xi}$ in $W_2^{1,2}(S_T)$ as $m \to \infty$. Now we can rewrite (2.55) as

$$\int_{\mathbb{R}^+} w_0(x)\hat{\xi}_m(x,0)\mathrm{d}x + \iint_{S_T} w\hat{\xi}_{mt}\mathrm{d}x\mathrm{d}t = \iint_{S_T} \mathcal{D}(w)\hat{\xi}_{mxx}\mathrm{d}x\mathrm{d}t,$$

with $\hat{\xi}_m \in C_0^\infty(\mathbb{R} \times [0,T))$, then letting $m \to \infty$, we deduce that for all $\hat{\xi} \in W_2^{1,2}(S_T)$ with $\hat{\xi}(\cdot, T) = 0$ and $\hat{\xi}(0, \cdot) = 0$, the difference $w - \tilde{w}$ satisfies

$$0 = \iint_{S_T} (w - \tilde{w})(\hat{\xi}_t + a\hat{\xi}_{xx})\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} (w_0 - \tilde{w}_0)\hat{\xi}(x,0)\mathrm{d}x, \tag{2.57}$$

where $a := \begin{cases} \dfrac{\mathcal{D}(w) - \mathcal{D}(\tilde{w})}{w - \tilde{w}} & w \neq \tilde{w}, \\ 0 & \text{otherwise.} \end{cases}$

Observe that $a \in L^\infty(S_T)$, now consider a sequence $\{a_n\}$ of smooth function such that $\frac{1}{n} \leq a_n \leq \|a\|_{L^\infty(S_T)} + \frac{1}{n}$ and $\dfrac{a_n - a}{\sqrt{a_n}} \to 0$ almost everywhere in S_T as $n \to \infty$.

Let $\tilde{\xi}_n \in W_2^{1,2}(S_T)$ be the solution of problem

$$\begin{cases} \lambda = \tilde{\xi}_{nt} + a_n\tilde{\xi}_{nxx}, & \text{in } S_T, \\ \tilde{\xi}_n(0,t) = 0, & t \in (0,T), \\ \tilde{\xi}_n(x,T) = 0, & x \in \mathbb{R}^+, \end{cases} \tag{2.58}$$

where $\lambda \in C_c^\infty(\mathbb{R}^+ \times [0, T))$.

The existence of a solution to this problem follows from standard parabolic theory, see for example [21, IV, Theorem 9.1]. We claim that the following estimates hold,

(i) $\|\tilde{\xi}_n\|_{L^\infty(S_T)} \leq C(\|\lambda\|_{L^\infty(S_T)}, T)$;

(ii) $\displaystyle\iint_{S_T} a_n|\tilde{\xi}_{nxx}|^2 \leq C(\lambda, T)$.

The maximum principle and a comparison of $\tilde{\xi}_n$ with the functions $\tilde{\xi}_n^+, \hat{\xi}_n^-$ defined by

$$\tilde{\xi}_n^+ = e^{\alpha(T-t)}, \quad \tilde{\xi}_n^- = -e^{\alpha(T-t)},$$

where $\alpha = e^T\|\lambda\|_{L^\infty(S_T)}$, gives (i).

To prove (ii), multiplying the equation of $\tilde{\xi}_n$ by $\tilde{\xi}_{nxx}$ and integrating over $\mathbb{R}^+ \times (t, T)$, we get

$$\begin{aligned}\int_t^T \int_{\mathbb{R}^+} \lambda\tilde{\xi}_{nxx} \mathrm{d}x\mathrm{d}t =& \int_t^T \int_{\mathbb{R}^+} \tilde{\xi}_{nt}\tilde{\xi}_{nxx} \mathrm{d}x\mathrm{d}t + \int_t^T \int_{\mathbb{R}^+} a_n(\tilde{\xi}_{nxx})^2 \mathrm{d}x\mathrm{d}t \\ =& \frac{1}{2}\int_{\mathbb{R}^+} (\tilde{\xi}_{nx})^2(t)\mathrm{d}x + \int_t^T \int_{\mathbb{R}^+} a_n(\tilde{\xi}_{nxx})^2 \mathrm{d}x\mathrm{d}t.\end{aligned}$$

We deduce from above that

$$\iint_{S_T} a_n(\tilde{\xi}_{nxx})^2 \mathrm{d}x\mathrm{d}t \leq \|\tilde{\xi}_n\|_{L^\infty(S_T)}\|\lambda_{xx}\|_{L^1(S_T)}.$$

Using $\tilde{\xi}_n$ as a test function of (2.57), we obtain

$$0 = \iint_{S_T} (w - \tilde{w})\left[\lambda + (a - a_n)\tilde{\xi}_{nxx}\right] \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} (w_0 - \tilde{w}_0)\tilde{\xi}_n(x, 0)\mathrm{d}x.$$

We deduce by Hölder's inequality and (ii) that

$$
\begin{aligned}
&\limsup_{n\to\infty} \left| \iint_{S_T} (w-\tilde{w})(a-a_n)\tilde{\xi}_{nxx} \right| \\
\leq &\limsup_{n\to\infty} \left\| \iint_{S_T} (w-\tilde{w})\frac{a-a_n}{\sqrt{a}_n}) \right\|_{L^2(S_T)} \left\| \sqrt{a}_n \tilde{\xi}_{nxx} \right\|_{L^2(S_T)} \\
=&0.
\end{aligned}
$$

Thus, in the limit $n \to \infty$, we get

$$
\iint_{S_T} (w-\tilde{w})\lambda \mathrm{d}x\mathrm{d}t \leq C(\|\lambda\|_{L^\infty(S_T)}, T) \int_{\mathbb{R}^+} (w_0 - \tilde{w}_0).
$$

Taking a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$, $\lambda_i \in C_c^\infty(S_T)$ with $\|\lambda_i\|_{L^\infty(S_T)} \leq 2$ and $\lambda_i \to \mathrm{sgn}(w-\tilde{w})$ almost everywhere, we obtain by letting $i \to \infty$

$$
\iint_{S_T} |w-\tilde{w}| \mathrm{d}x\mathrm{d}t \leq C(T) \int_{\mathbb{R}^+} |w_0 - \tilde{w}_0| \mathrm{d}x.
$$

□

By Theorem 2.31, Theorem 2.32 and Theorem 2.33, we obtain the following.

Theorem 2.34. *Let $\varepsilon \geq 0$. Then there exists a unique solution w of the limit problem (2.52).*

Chapter 3

The half-line case: self-similar solutions for the limit problems

3.1 Self-similar solutions for the limit problems

The existence and uniqueness of the weak solution of problem (2.52) are guaranteed by Theorem 2.34. In the following, we will prove that (2.52) has a self-similar solution, which therefore be this unique weak solution. Our strategy takes advantages of some ideas from [10] and [16].

We first state a free-boundary problem, including interface conditions, that is satisfied by the solution w of (2.52) under some regularity assumptions and conditions on the form of the free boundary.

Theorem 3.1. *Let w be the unique weak solution of problem (2.52). Suppose that there exists a function $\beta : [0,T] \to \mathbb{R}^+$ such that for each $t \in [0,T]$,*

$$w(x,t) > 0 \text{ if } x < \beta(t) \quad \text{and} \quad w(x,t) < 0 \text{ if } x > \beta(t).$$

Then if $t \mapsto \beta(t)$ is sufficiently smooth and the functions $u := w^+$ and $v := -w^-$ are smooth up to $\beta(t)$, the functions u, v satisfy

$$\begin{cases} u_t = \phi(u)_{xx}, & \text{in } (x,t) \in S_T : x < \beta(t), \\ v_t = \varepsilon\phi(v)_{xx}, & \text{in } (x,t) \in S_T : x > \beta(t), \\ \langle \phi(u) \rangle = \varepsilon \langle \phi(v) \rangle = 0, & \text{on } \Gamma_T := \{(x,t) \in S_T : x = \beta(t)\}, \\ \langle v \rangle \beta'(t) = \langle \phi(u)_x - \varepsilon\phi(v)_x \rangle, & \text{on } \Gamma_T := \{(x,t) \in S_T : x = \beta(t)\}, \\ u = U_0, & \text{on } \{0\} \times [0,T], \\ u(\cdot,0) = u_0^\infty(\cdot),\ v(\cdot,0) = v_0^\infty(\cdot), & \text{in } \mathbb{R}^+, \end{cases} \tag{3.1}$$

where $\langle \cdot \rangle$ denotes the jump across $\beta(t)$ from $\{x < \beta(t)\}$ to $\{x > \beta(t)\}$,

$$\langle \alpha \rangle := \lim_{x \searrow \beta(t)} \alpha(x,t) - \lim_{x \nearrow \beta(t)} \alpha(x,t),$$

and $\beta'(t)$ denotes the speed of propagation of the free boundary $\beta(t)$.

Proof. We recall that (u,v) satisfies

$$\iint_{S_T} (u-v)\xi_t \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^+} (u_0^\infty - v_0^\infty)\xi(x,0)\mathrm{d}x = \iint_{S_T} (\phi(u) - \varepsilon\phi(v))_x \xi_x \mathrm{d}x\mathrm{d}t,$$

for all $\xi \in \mathcal{F}_T$.

Next we consider the time derivative term, and find that

$$\iint_{S_T} (u-v)\xi_t \mathrm{d}x\mathrm{d}t = \iint_{S_T} u\xi_t \mathrm{d}x\mathrm{d}t - \iint_{S_T} v\xi_t \mathrm{d}x\mathrm{d}t, \tag{3.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\beta(t)} u\xi \mathrm{d}x &= \int_0^{\beta(t)} (u_t\xi + u\xi_t)\mathrm{d}x + \lim_{x \nearrow \beta(t)} u(x,t)\xi(\beta(t),t)\beta'(t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\beta(t)}^\infty v\xi \mathrm{d}x &= \int_{\beta(t)}^\infty (v_t\xi + v\xi_t)\mathrm{d}x - \lim_{x \searrow \beta(t)} v(x,t)\xi(\beta(t),t)\beta'(t), \end{aligned}$$

from which we obtain

$$\iint_{S_T}(u-v)\xi_t \mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^+}(u_0^\infty - v_0^\infty)\xi(x,0)\mathrm{d}x$$
$$=-\int_0^T\int_0^{\beta(t)} u_t\xi \mathrm{d}x\mathrm{d}t+\int_0^T\int_{\beta(t)}^\infty v_t\xi \mathrm{d}x\mathrm{d}t+\int_0^T \langle u-v\rangle \xi(\beta(t),t)\beta'(t)\mathrm{d}t. \tag{3.3}$$

Analysis of the diffusion term then gives

$$\begin{aligned}\int_0^T\int_0^{\beta(t)}\phi(u)_x\xi_x\mathrm{d}x\mathrm{d}t=&\int_0^T \lim_{x\nearrow\beta(t)}\phi(u)_x\xi(\beta(t),t)\mathrm{d}t\\ &-\int_0^T\int_0^{\beta(t)}\phi(u)_{xx}\xi\mathrm{d}x\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\varepsilon\int_0^T\int_{\beta(t)}^\infty\phi(v)_x\xi_x\mathrm{d}x\mathrm{d}t=&-\varepsilon\int_0^T \lim_{x\searrow\beta(t)}\phi(v)_x\xi(\beta(t),t)\mathrm{d}t\\ &-\varepsilon\int_0^T\int_{\beta(t)}^\infty\phi(v)_{xx}\xi\mathrm{d}x\mathrm{d}t,\end{aligned}$$

and then combining these two equations yields

$$\begin{aligned}&\iint_{S_T}(\phi(u)_x-\varepsilon\phi(v)_x)\xi_x\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\langle-\phi(u)_x+\varepsilon\phi(v)_x\rangle\xi(\beta(t),t)\mathrm{d}t-\int_0^T\int_0^{\beta(t)}\phi(u)_{xx}\xi\mathrm{d}x\mathrm{d}t\\ &\quad+\varepsilon\int_0^T\int_{\beta(t)}^\infty\phi(v)_{xx}\xi\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.4}$$

Therefore the computations (3.2), (3.3) and (3.4) yield

$$\begin{aligned}&\int_0^T\langle -u+v\rangle\xi(\beta(t),t)\beta'(t)\mathrm{d}t+\int_0^T\langle-\phi(u)_x+\varepsilon\phi(v)_x\rangle\xi(\beta(t),t)\mathrm{d}t\\ &=\int_0^T\int_0^{\beta(t)}\left(\phi(u)_{xx}-u_t\right)\xi\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\beta(t)}^\infty\left(\varepsilon\phi(v)_{xx}-v_t\right)\xi\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.5}$$

for all $\xi\in\mathcal{F}_T$.

Now by using test functions with suitable support, namely $\operatorname{supp}\xi\subset\big\{(x,t)\in S_T\big|\ x<\beta(t)\big\}$ and $\operatorname{supp}\xi\subset\big\{(x,t)\in S_T\big|\ x>\beta(t)\big\}$, we obtain

$$u_t = \phi(u)_{xx}, \qquad \text{in } \{(x,t) \in S_T : x < \beta(t)\},$$

$$v_t = \varepsilon\phi(v)_{xx}, \qquad \text{in } \{(x,t) \in S_T : x > \beta(t)\},$$

and the remaining terms in (3.5) allow us to conclude that

$$\int_0^T \langle -u + v\rangle \xi(\beta(t), t)\beta'(t)\mathrm{d}t + \int_0^T \langle -\phi(u)_x + \varepsilon\phi(v)_x\rangle \xi_x(\beta(t), t)\mathrm{d}t = 0,$$

which gives

$$\langle -u + v\rangle\beta'(t) + \langle -\phi(u)_x + \varepsilon\phi(v)_x\rangle = 0 \quad \text{on } \Gamma_T := \{(x,t) \in S_T : x = \beta(t)\}. \tag{3.6}$$

Now we know that $\mathcal{D}(w)$ is a continuous function of x for almost every $t \in [0, T]$, since $\mathcal{D}(w) \in \mathcal{D}(\hat{w}) + L^2(0, T; W^{1,2}(\mathbb{R}^+))$ by Definition 2.30 (ii). So $\langle \mathcal{D}(w)\rangle = 0$, which implies

$$-\lim_{x\searrow\beta(t)} \varepsilon\phi(-w^-) - \lim_{x\nearrow\beta(t)} \phi(w^+) = -\lim_{x\searrow\beta(t)} \varepsilon\phi(v) - \lim_{x\nearrow\beta(t)} \phi(u) = 0.$$

Therefore we get

$$\langle\phi(u)\rangle = \varepsilon\langle\phi(v)\rangle = 0. \tag{3.7}$$

Moreover, since $\phi \in C^2(\mathbb{R})$ is strictly increasing, $u(\cdot, t)$ is continuous across $\beta(t)$ and if $\varepsilon > 0$, $v(\cdot, t)$ is also continuous across $\beta(t)$, so that

$$\langle u\rangle = 0 \quad \text{if } \varepsilon \geq 0, \tag{3.8}$$

$$\langle v\rangle = 0 \quad \text{if } \varepsilon > 0. \tag{3.9}$$

Then (3.6) and the fact that $\langle u\rangle = 0$ imply that

$$\langle v\rangle\beta'(t) = \langle\phi(u)_x - \varepsilon\phi(v)_x\rangle, \quad \text{on } \Gamma_T. \tag{3.10}$$

$\square$

The following two limit problems are obtained by interpreting the interface conditions on $\beta(t)$.

Corollary 3.2. *Let w and $\beta : [0,t] \to \mathbb{R}^+$ satisfy the hypotheses of theorem 3.1. Then the functions $u := w^+$, $v := -w^-$ satisfy one of limit problems depending on whether $\varepsilon > 0$ or $\varepsilon = 0$. If $\varepsilon > 0$, then*

$$\begin{cases} u_t = \phi(u)_{xx}, & \text{in } \{(x,t) \in S_T : x < \beta(t)\}, \\ v = 0, & \text{in } \{(x,t) \in S_T : x < \beta(t)\}, \\ v_t = \varepsilon\phi(v)_{xx}, & \text{in } \{(x,t) \in S_T : x > \beta(t)\}, \\ u = 0, & \text{in } \{(x,t) \in S_T : x > \beta(t)\}, \\ \lim\limits_{x \nearrow \beta(t)} u(x,t) = 0 = \lim\limits_{x \searrow \beta(t)} v(x,t) & \text{for each } t \in [0,T], \\ \lim\limits_{x \nearrow \beta(t)} \phi[u(x,t)]_x = -\varepsilon \lim\limits_{x \searrow \beta(t)} \phi[v(x,t)]_x & \text{for each } t \in [0,T], \\ u = U_0, & \text{on } \{0\} \times [0,T], \\ u(\cdot,0) = u_0^\infty(\cdot),\ v(\cdot,0) = v_0^\infty(\cdot), & \text{in } \mathbb{R}^+, \end{cases} \tag{3.11}$$

whereas if $\varepsilon = 0$ and we suppose additionally that $\beta(0) = 0$ and $t \mapsto \beta(t)$ is *a non-decreasing function, then*

$$\begin{cases} u_t = \phi(u)_{xx}, & \text{in } \{(x,t) \in S_T : x < \beta(t)\}, \\ v = 0, & \text{in } \{(x,t) \in S_T : x < \beta(t)\}, \\ v = V_0, & \text{in } \{(x,t) \in S_T : x > \beta(t)\}, \\ u = 0, & \text{in } \{(x,t) \in S_T : x > \beta(t)\}, \\ \lim\limits_{x \nearrow \beta(t)} u(x,t) = 0 & \text{for each } t \in [0,T], \\ V_0 \beta'(t) = -\lim\limits_{x \nearrow \beta(t)} \phi[u(x,t)]_x & \text{for each } t \in [0,T], \\ u = U_0, & \text{on } \{0\} \times [0,T], \\ u(\cdot,0) = u_0^\infty(\cdot),\ v(\cdot,0) = v_0^\infty(\cdot), & \text{in } \mathbb{R}^+, \end{cases} \tag{3.12}$$

where $\beta'(t)$ denotes the speed of propagation of the free boundary $\beta(t)$.

Proof. We have $\langle u \rangle = 0$ from (3.8). From (3.9), we know that if $\varepsilon > 0$, $\langle v \rangle = 0$, whereas if $\varepsilon = 0$, $v(\cdot,t)$ jumps across $\beta(t)$. The fact that $v_t = 0$ in $\{(x,t) \in S_T : x > \beta(t)\}$ together with the initial condition that $v_0^\infty(x) = V_0$ if $x > 0$ give the result that $v(x,t) = V_0$ for all $x \geq \beta(t)$, since $\beta(0) = 0$ and $t \mapsto \beta(t)$ is a non-decreasing function. It follows that if $\varepsilon = 0$

$$\langle v \rangle = V_0 - 0 = V_0 \text{ for all } t \in [0,T].$$

The normal derivative condition (3.10) implies that if $\varepsilon > 0$, then $\langle \phi(u)_x - \varepsilon\phi(v)_x \rangle = 0$, so that

$$\lim_{x \nearrow \beta(t)} \phi[u(x,t)]_x = -\varepsilon \lim_{x \searrow \beta(t)} \phi[v(x,t)]_x.$$

On the other hand, if $\varepsilon = 0$, then

$$V_0 \beta'(t) = -\lim_{x \nearrow \beta(t)} \phi(u)_x.$$

$\square$

Next we will prove that if we have a self-similar solution of (3.13), then it is a weak solution of (2.52) in the sense of Definition 2.30. We will then prove the existence of the self-similar solution of (3.13) by a two-parameter method in Section 3.3.

Theorem 3.3. *The unique weak solution w of problem (2.52) with $\varepsilon > 0$ has a self-similar form. There exists a function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and a constant $a \in \mathbb{R}^+$ such that*

$$w(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \ (x,t) \in S_T \text{ and } \beta(t) = a\sqrt{t}, \ t \in [0,T].$$

Denoting $\eta = \dfrac{x}{\sqrt{t}}$, f satisfies the system

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', & \text{if } \eta < a, \\ -\dfrac{1}{2}\eta f'(\eta) = [\varepsilon\phi'(-f(\eta))f'(\eta)]', & \text{if } \eta > a, \\ f(0) = U_0, \quad \lim\limits_{\eta\to\infty} f(\eta) = -V_0, \\ \lim\limits_{\eta\nearrow a} f(\eta) = 0 = -\lim\limits_{\eta\searrow a} f(\eta), \\ \lim\limits_{\eta\nearrow a} \phi'(f(\eta))f'(\eta) = \varepsilon \lim\limits_{\eta\searrow a} \phi'(-f(\eta))f'(\eta), \end{cases} \tag{3.13}$$

where a prime denotes differentiation with respect to η.

Proof. Note first that from Theorem 2.34, we know that if $w(x,t) = f\left(\frac{x}{\sqrt{t}}\right)$ is a weak solution of (2.52), then it is unique. We therefore need to show that a solution to (3.13) exists, which will be postponed to Section 3.3, and that if f satisfies (3.13), then it is a weak solution of (2.52), that is, it satisfies Definition 2.30.

Now the weak solution of (2.52) satisfies

$$\iint_{S_T} w\xi_t \mathrm{d}x\mathrm{d}t - \iint_{S_T} \mathcal{D}(w)_x \xi_x \mathrm{d}x\mathrm{d}t = V_0 \int_{\mathbb{R}^+} \xi(x,0)\mathrm{d}x,$$

where $\xi \in \mathcal{F}_T$. If we write $w(x,t) = f\left(\frac{x}{\sqrt{t}}\right)$, then since $\xi(\cdot, T) = 0$, the left-hand side becomes

$$\begin{aligned}
&\iint_{S_T} f\left(\frac{x}{\sqrt{t}}\right)\xi_t \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^{a\sqrt{t}} t^{-\frac{1}{2}}\phi'\left(f\left(\frac{x}{\sqrt{t}}\right)\right) f'\left(\frac{x}{\sqrt{t}}\right)\xi_x \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{a\sqrt{t}}^{\infty} t^{-\frac{1}{2}}\varepsilon\phi'\left(-f\left(\frac{x}{\sqrt{t}}\right)\right) f'\left(\frac{x}{\sqrt{t}}\right)\xi_x \mathrm{d}x\mathrm{d}t \\
&= V_0 \int_{\mathbb{R}^+} \xi(x,0)\mathrm{d}x + \frac{1}{2}\iint_{S_T} x t^{-\frac{3}{2}} f'\left(\frac{x}{\sqrt{t}}\right)\xi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_0^{a\sqrt{t}} t^{-1}(\phi'(f)f')'\xi_x \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{a\sqrt{t}}^{\infty} t^{-1}\varepsilon(\phi'(-f)f')'\xi_x \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

since $\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta) = \varepsilon \lim_{\eta \searrow a} \phi'(-f(\eta))f'(\eta)$.

Since f satisfies (3.13), we therefore have

$$\begin{aligned}
&\iint_{S_T} w\xi_t \mathrm{d}x\mathrm{d}t - \iint_{S_T} \mathcal{D}(w)_x \xi_x \mathrm{d}x\mathrm{d}t \\
&= V_0 \int_{\mathbb{R}^+} \xi(x,0)\mathrm{d}x + \frac{1}{2}\iint_{S_T} x t^{-\frac{3}{2}} f'\left(\frac{x}{\sqrt{t}}\right)\xi \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_0^{a\sqrt{t}} t^{-1}(\phi'(f)f')'\xi_x \mathrm{d}x\mathrm{d}t + \int_0^T \int_{a\sqrt{t}}^{\infty} t^{-1}\varepsilon(\phi'(-f)f')'\xi_x \mathrm{d}x\mathrm{d}t \\
&= V_0 \int_{\mathbb{R}^+} \xi(x,0)\mathrm{d}x.
\end{aligned}$$

and hence f satisfies Definition 2.30 (iii).

Next we will prove that f satisfies Definition 2.30 (ii). First, we introduce some lemmas that will be used to prove $\phi(U_0) - \phi(f) \in L^2((0,a))$, $\phi'(f) \in L^2((0,a))$, $\phi(V_0) - \phi(-f) \in L^2((a,\infty))$ and $\phi'(-f) \in L^2((a,\infty))$. These lemmas will also be useful later.

We define

$$\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta). \tag{3.14}$$

Note that

$$\gamma = -\varepsilon \lim_{\eta \searrow a} \phi'(-f(\eta))f'(\eta) \quad \text{when } \varepsilon > 0,$$

which we can obtain from the free boundary condition of Problem (3.13).

Next, we prove the monotonicity of f.

Lemma 3.4. *Suppose $\varepsilon > 0$. If f satisfies (3.13), then $f'(\eta) < 0$ for all $\eta \neq a$.*

Proof. Suppose f is not monotonic, then there exists $\eta_0 \neq a$ such that $f'(\eta_0) = 0$, denote $f_0 := f(\eta_0) \neq 0$. Then defining the function by $g : \mathbb{R} \to \mathbb{R}^+$

$$g(\eta) = f_0 > 0, \quad \text{for all } \eta \in \mathbb{R},$$

we have that g satisfies

$$\begin{aligned} -\frac{1}{2}\eta g'(\eta) &= [\phi'(g(\eta))g'(\eta)]' \\ g(\eta_0) = f_0, &\quad g'(\eta_0) = 0, \end{aligned}$$

but also

$$\begin{aligned} -\frac{1}{2}\eta f'(\eta) &= [\phi'(f(\eta))f'(\eta)]' \\ f(\eta_0) = f_0, &\quad f'(\eta_0) = 0. \end{aligned}$$

By Picard's theorem and the uniqueness, it follows that $f = g$ either for all $\eta < a$ or $\eta > a$, depending on whether $f(\eta_0) > 0$ or $f(\eta_0) < 0$. But this contradict the boundary conditions in (3.13), so we know that f must be monotonically decreasing. $\square$

Next, we prove that γ defined in (3.14) is strictly positive when $\varepsilon > 0$. In fact, γ is also strictly positive when $\varepsilon = 0$, see Corollary 3.12.

Lemma 3.5. *Suppose $\varepsilon > 0$. Let f be a solution of (3.13), then $\gamma > 0$.*

Proof. If $\gamma \leq 0$, then integrating the equation for $\eta > a$ in (3.13) from a to η yields

$$-\frac{1}{2}\int_a^\eta sf'(s)\mathrm{d}s = \varepsilon\phi'(-f(\eta))f'(\eta) + \gamma. \tag{3.15}$$

The left-hand side of (3.15) is positive since $f' < 0$ by Lemma 3.4 whereas the right-hand side of (3.15) is negative since $f' < 0$ and $\gamma \leq 0$. Therefore, it follows that $\gamma > 0$ by the contradiction. $\square$

It is clear that $f \in L^\infty(S_T)$, so it satisfies Definition 2.30 (i). Now we prove some properties of f'.

Lemma 3.6. *Suppose $\varepsilon > 0$. If f satisfies (3.13), then for $\eta > a$, f' is monotonically increasing in η.*

Proof. The results follow from the equation for f when $\eta > a$

$$-\frac{1}{2}\eta f'(\eta) = -\varepsilon\phi''(-f(\eta))(f'(\eta))^2 + \varepsilon\phi'(-f(\eta))f''(\eta).$$

The left-hand side is positive since $0 < a < \eta$ and the first term of right-hand side is negative. Therefore f'' must be positive. $\square$

In the following, we prove a bound for $-f'$ when $\eta > a$.

Lemma 3.7. *Suppose $\varepsilon > 0$. If f satisfies (3.13), then we have for $\eta > a$*

$$-f'(\eta) \leq \frac{4\gamma}{\eta^2 - a^2}, \tag{3.16}$$

and hence, in particular,

$$\lim_{\eta\to\infty} f'(\eta) = 0. \tag{3.17}$$

Proof. Integrating the equation of f for $\eta > a$ from a to η, we have

$$-\frac{1}{2}\int_a^\eta sf'(s)\mathrm{d}s = \varepsilon\phi'(-f(\eta))f'(\eta) + \gamma.$$

Since $f' < 0$ is increasing and the right-hand side is positive, then

$$-\frac{1}{4}f'(\eta)(\eta^2 - a^2) \leq \varepsilon\phi'(-f(\eta))f'(\eta) + \gamma \leq \gamma.$$

Therefore we obtain

$$-f'(\eta) \leq \frac{4\gamma}{\eta^2 - a^2}.$$

If we choose $\eta > a + 1$ then

$$-f'(\eta) < \frac{4\gamma}{2\eta - 1}, \tag{3.18}$$

which vanishes as $\eta \to \infty$. □

The following lemma proves that $\phi(-f)$ converges to $\phi(V_0)$ exponentially.

Lemma 3.8. *Suppose $\varepsilon > 0$. If f satisfies (3.13), then for $\eta > a$ we have*

$$0 \leq \phi(V_0) - \phi(-f(\eta)) \leq C\int_\eta^\infty e^{\frac{-s^2}{4\varepsilon\phi'(V_0)}}\mathrm{d}s, \tag{3.19}$$

where $C = \gamma e^{\frac{a^2}{4\varepsilon\phi'(V_0)}}$.

Proof. Denote $N = \phi'(V_0)$. We have $\phi'(-f) < \phi'(V_0)$ since f is monotonically decreasing by Lemma 3.4 and ϕ' is increasing. Then we get directly from the equation of f for $\eta > a$ that

$$\frac{\eta}{2\varepsilon N}[\phi(-f)]' \leq -[\phi(-f)]'',$$

then multiplying by $e^{\frac{\eta^2}{4\varepsilon N}}$, we get

$$\left\{e^{\frac{\eta^2}{4\varepsilon N}}[\phi(-f)]'\right\}' \leq 0,$$

integrating from a to η yields

$$[\phi(-f(\eta))]' \leq Ce^{\frac{-\eta^2}{4\varepsilon N}}, \tag{3.20}$$

by Lemma 3.7, where $C = \gamma e^{\frac{a^2}{4\varepsilon N}} > 0$. Then integrating from η to ∞ we get

$$\phi(V_0) - \phi(-f(\eta)) \leq C \int_{\eta}^{\infty} e^{\frac{-s^2}{4\varepsilon\phi'(V_0)}} \mathrm{d}s.$$

$\square$

We get the following corollary immediately from (3.20).

Corollary 3.9. *Suppose $\varepsilon > 0$. If f satisfies (3.13), then for $a < \eta < a+1$ we have*

$$0 \leq -\phi'(-f(\eta))f'(\eta) \leq D, \tag{3.21}$$

where $D = e^{\frac{(a+1)^2}{4\varepsilon\phi'(V_0)}} \phi'(-f(a+1))f'(a+1)$.

The following lemma yields a bound for $-\phi'(f(\eta))f'(\eta)$ for $\eta < a$.

Lemma 3.10. *If f satisfies (3.13), then for $0 \leq \eta < a$ we have*

$$0 \leq -\phi'(f(\eta))f'(\eta) \leq \hat{C}, \tag{3.22}$$

where $\hat{C} = -\phi'(f(0))f'(0)$.

Proof. Denote $\hat{N} = \phi'(U_0)$, similarly to the proof of Lemma 3.8, we have

$$\left\{ e^{\frac{\eta^2}{4\hat{N}}} [\phi(f)]' \right\}' \geq 0,$$

and then integrating from 0 to η we get

$$0 \leq -\phi'(f(\eta))f'(\eta) \leq \hat{C}e^{\frac{-\eta^2}{4\hat{N}}} < \hat{C}. \tag{3.23}$$

$\square$

Now we have $\phi(U_0) - \phi(f) \in L^2((0,a))$, since Lemma 3.4 implies $f(\eta) < U_0$ for $\eta < a$, and $-(\phi(f))' \in L^2((0,a))$ by Lemma 3.10. From Lemma 3.4 and Lemma 3.8, we know that $\phi(V_0) - \phi(-f) \in L^2((a,\infty))$, since

$$\int_a^{a+1} |\phi(V_0) - \phi(-f(s))|^2 \mathrm{d}s < (\phi(V_0))^2,$$
$$\int_{a+1}^{\infty} |\phi(V_0) - \phi(-f(s))|^2 \mathrm{d}s < \int_{a+1}^{\infty} e^{\frac{-s}{4\varepsilon\phi'(V_0)}} \mathrm{d}s < 4\varepsilon\phi'(V_0) e^{\frac{-a-1}{4\varepsilon\phi'(V_0)}}.$$

Finally, $-(\phi(-f))' \in L^2((a,\infty))$ is implied by (3.18) and Corollary 3.9. It therefore follows that if f satisfies (3.13), then it satisfies Definition 2.30 (ii) that is

$$\mathcal{D}(f) \in \mathcal{D}(\hat{w}) + L^2(0,T; W_0^{1,2}(\mathbb{R}^+)),$$

since by changing variables

$$\int_0^T \int_0^{a\sqrt{t}} |\phi(U_0) - \phi(w)(x,t)|^2 \mathrm{d}x\mathrm{d}t = \int_0^T \sqrt{t} \int_0^a |\phi(U_0) - \phi(f(s))|^2 \mathrm{d}s\mathrm{d}t \leq \frac{2}{3} C T^{\frac{3}{2}},$$

and

$$\int_0^T \int_0^{a\sqrt{t}} |\phi(w)_x|^2 \mathrm{d}x\mathrm{d}t = \int_0^T \frac{1}{\sqrt{t}} \int_0^a |\phi'(f) f'|^2 \mathrm{d}s\mathrm{d}t \leq 2CT^{\frac{1}{2}},$$

where C is a constant. Similar calculations can be done on $(a\sqrt{t}, \infty) \times (0,T)$. Hence f satisfies Definition 2.30. It remains to prove the existence of solution of Problem (3.13), which is done in Theorem 3.30 in Section 3.3. $\square$

Similarly, we can prove that if we have a self-similar solution of (3.24) when $\varepsilon = 0$, then it is a weak solution of (2.52) in the sense of Definition 2.30.

Theorem 3.11. *The unique weak solution w of problem (2.52) with $\varepsilon = 0$ has a self-similar form. There exists a function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and a constant $a \in \mathbb{R}^+$ such that*

$$w(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \ (x,t) \in S_T \text{ and } \beta(t) = a\sqrt{t}, \ t \in [0,T].$$

Denote $\eta = \dfrac{x}{\sqrt{t}}$, f satisfies the system

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', & \text{if } \eta < a, \\ f(\eta) = -V_0, & \text{if } \eta > a, \\ f(0) = U_0, \\ \lim\limits_{\eta \nearrow a} f(\eta) = 0, \\ \lim\limits_{\eta \nearrow a} \phi'(f(\eta))f'(\eta) = -\dfrac{aV_0}{2}, \end{cases} \tag{3.24}$$

where a prime denotes differentiation with respect to η.

Proof. When $\varepsilon = 0$, we can only consider f for $\eta < a$, since $f(\eta) = -V_0$ for $\eta > a$. The proof of the fact that a solution of (3.24) yields a weak solution of (2.52) is similar as the proof of Theorem 3.3 and the existence of solution for problem (3.24) is proved in Theorem 3.34 by the shooting method. $\square$

The following corollary follows from the fact that $\gamma = \dfrac{aV_0}{2}$ when $\varepsilon = 0$, which is directly from the free boundary conditions on (3.24).

Corollary 3.12. *Suppose $\varepsilon = 0$. Let f be a solution of (3.24), then $\gamma = \dfrac{aV_0}{2}$ is positive.*

3.2 Self-similar solutions with $\varepsilon > 0$

We begin with some discussion of earlier work on self-similar solutions for nonlinear diffusion problems. Let $k(s)$ be continuous with $k(0) = 0$ and

$k(s) > 0$ as $s > 0$. We introduce the notation $k(s)$ for ease of comparison with previous papers [2, 7], but, k clearly plays the same role as ϕ' plays in this thesis. Then the solutions of the nonlinear diffusion equation

$$u_t = (k(u)_x)_x$$

can be studied in self-similar form

$$u(x,t) = f(\eta), \quad \eta = xt^{-\frac{1}{2}},$$

and f satisfies the equation

$$(k(f)f')' + \frac{1}{2}\eta f' = 0. \tag{3.25}$$

In [2], Atkinson and Peletier proved the existence and uniqueness of a self-similar solution $f(\eta)$ which satisfies (3.25) for $0 < \eta < a$, where $a > 0$, under the boundary conditions $f(0) = U$, $\lim_{\eta \to a} f(\eta) = 0$ and $\lim_{\eta \to a} k(f(\eta))f'(\eta) = 0$. They consider two cases in describing the dependence of a and U,

A. $\displaystyle\int_1^\infty \frac{k(s)}{s} \mathrm{d}s = \infty;$

B. $\displaystyle\int_1^\infty \frac{k(s)}{s} \mathrm{d}s < \infty.$

They found that as $U \to \infty$, $a = a(U)$ tends to infinity in Case A whereas $a(U)$ tends to a finite limit in Case B. In this thesis, we only consider the case when $\displaystyle\int_1^\infty \frac{k(s)}{s} \mathrm{d}s = \infty$. The proof in [2] depends on a discussion of the function $b(a)$, which is defined as the value at $\eta = 0$ of the solution $f(\eta) = f(\eta; a)$ of (3.25) with boundary conditions $\lim_{\eta \to a} f(\eta) = 0$ and $\lim_{\eta \to a} k(f(\eta))f'(\eta) = 0$. A similar problem in an unbounded interval $0 < \eta < \infty$ with boundary conditions $f(0) = U$ and $\lim_{\eta \to \infty} f(\eta) = 0$ is studied in [7] by Craven and Peletier. Note that in [7], $f(\eta) > 0$ for all $\eta > 0$. The paper [7] proved the existence and uniqueness of a weak solution by a shooting method where the initial value problem $f(0) = U$, $f'(0) = \beta$ is considered. We will adapt ideas of s-tudying the function $b(a)$ and shooting methods from [2, 7] to prove existence of self-similar solutions in this thesis.

Now recall that $\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta)$. Note that by Lemma 3.5 and Corollary 3.12, γ is strictly positive, which contrasts with [2] where $\gamma = 0$. We will study the existence of solution f that satisfies (3.13) with given boundary conditions by splitting it into two parts: $\eta < a$ where $f(\eta)$ is positive, and $\eta > a$ where $f(\eta)$ is negative. Then we will discuss the existence and properties of $\lim_{\eta \to 0} f(\eta)$ and $\lim_{\eta \to \infty} f(\eta)$. These results will be used to study $b(a, \gamma)$, the value at $\eta \to 0$ of the solution $f(\eta) = f(\eta; a, \gamma)$, and $d(a, \gamma)$, the value at $\eta \to \infty$ of the solution $f(\eta) = f(\eta; a, \gamma)$, and also to implement a two-parameter shooting method in Section 3.3.

First we consider f that satisfies the equation

$$-\frac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', \quad 0 < \eta < a. \tag{3.26}$$

At the boundaries we seek a solution that satisfies

$$f(0) = U_0, \tag{3.27}$$

$$\lim_{\eta \nearrow a} f(\eta) = 0, \quad \lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta) = -\gamma. \tag{3.28}$$

3.2.1 Solution in left-neighbourhood of $\eta = a$

We start by proving the local existence of a positive solution of (3.26) in a left-neighbourhood of $\eta = a$, which satisfies the boundary conditions (3.28).

First we prove some estimates that will be used later.

Lemma 3.13. *Suppose $a > 0$ and choose $a_1 < a$. If f satisfies (3.26) and* *the boundary conditions (3.28), then*

$$\int_0^{f(a_1)} \frac{\phi'(f)}{\frac{2\gamma}{a}+f}\mathrm{d}f \le \frac{1}{2}a^2. \tag{3.29}$$

Proof. Integration of (3.26) from η to a yields

$$-\frac{1}{2}\int_\eta^a sf'(s)\mathrm{d}s = -\gamma - \phi'(f(\eta))f'(\eta).$$

In view of Lemma 3.4, $f'(\eta)$ is negative, then since $a > 0$

$$|\phi'(f(\eta))f'(\eta)| \le \gamma - \frac{1}{2}a\int_\eta^a f'(s)\mathrm{d}s.$$

Thus, for any a_2 such that $a_1 < a_2 < a$, we have

$$\int_{a_1}^{a_2} \frac{\phi'(f(\eta))|f'(\eta)|}{\frac{2\gamma}{a}+f(\eta)}\mathrm{d}\eta \le \frac{1}{2}a(a_2 - a_1).$$

Since f is monotonic, this implies

$$\int_{f(a_2)}^{f(a_1)} \frac{\phi'(f)}{\frac{2\gamma}{a}+f}\mathrm{d}f \le \frac{1}{2}a^2,$$

and (3.29) follows by letting a_2 tend to a, so that $f(a_2)$ tends to zero. $\square$

Next we prove the existence and uniqueness of the local solution by using a method inspired by [2, Lemmas 4 and 5].

Lemma 3.14. *For given a and γ, there exists $\delta > 0$ such that for $\eta \in (a-\delta, a)$, equation (3.26) has a unique solution which is positive and satisfies the boundary condition (3.28).*

Proof. It is convenient to start by supposing that such a solution exists in a left-neighbourhood of $\eta = a$. Integrating (3.26) from η to a yields

$$-\frac{1}{2}\int_\eta^a sf'(s)\mathrm{d}s = -\gamma - \phi'(f(\eta))f'(\eta), \tag{3.30}$$

and hence

$$\frac{1}{f'(\eta)} = \frac{2\phi'(f)}{\int_\eta^a sf'(s)\mathrm{d}s - 2\gamma}. \tag{3.31}$$

Since f is monotonic, with non-vanishing derivative, we can treat η as a function of f, writing $\eta = \sigma(f)$. Then (3.31) takes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}f} = \frac{-2\phi'(f)}{\int_0^f \sigma(s)\mathrm{d}s + 2\gamma},$$

and $\sigma(f)$ is a solution of this integro-differential equation which satisfies the initial condition $\sigma(0) = a$ and is defined and continuous on an interval $[0, \hat{f}]$ for some $\hat{f} > 0$ and continuously differentiable on $(0, \hat{f})$. An integration gives

$$\sigma(f) = a - 2\int_0^f \frac{\phi'(\theta)}{\int_0^\theta \sigma(s)\mathrm{d}s + 2\gamma}\mathrm{d}\theta, \tag{3.32}$$

and if we set

$$\tau(f) = 1 - \frac{\sigma(f)}{a} = 1 - \frac{\eta}{a},$$

then (3.32) becomes

$$\tau(f) = 2a^{-2}\int_0^f \frac{\phi'(\theta)}{\int_0^\theta [1-\tau(s)]\mathrm{d}s + \frac{2\gamma}{a}}\mathrm{d}\theta. \tag{3.33}$$

If the solution of (3.33) is unique, the corresponding solution of equation (3.26) is also unique.

Now we prove (3.33) has a unique solution on $[0, \mu]$ for some $\mu > 0$, from which it follows that (3.26) has a unique solution on $(a - \delta, a)$.

Lemma 3.15. *There exists $\mu > 0$ such that (3.33) has a unique continuous solution in $0 \leq f \leq \mu$, which is such that $\tau(0) = 0$ and $\tau(f) > 0$ if $0 < f \leq \mu$.*

Proof. With μ to be chosen later, we denote by X the set of continuous functions $\tau(f)$ defined on $[0, \mu]$, satisfying $0 \leq \tau(f) \leq \frac{1}{2}$. We denote by $\| \,\bullet\, \|$ the supremum norm on X. Then X is a complete metric space. On X we introduce the map

$$\begin{aligned} M(\tau)(f) =& 2a^{-2} \int_0^f \frac{\phi'(\theta)}{\int_0^\theta [1 - \tau(s)]\mathrm{d}s + \frac{2\gamma}{a}} \mathrm{d}\theta \\ \leq& 4a^{-2} \int_0^\mu \frac{\phi'(\theta)}{\theta + \frac{4\gamma}{a}} \mathrm{d}\theta. \end{aligned}$$

It is clear that $M(\tau)(f)$ is well-defined, non-negative and continuous. Moreover, $M(\tau)(f) \leq \frac{1}{2}$ if

$$4a^{-2} \int_0^\mu \frac{\phi'(\theta)}{\theta + \frac{4\gamma}{a}} \mathrm{d}\theta \leq \frac{1}{2}. \tag{3.34}$$

Therefore, if μ is chosen small enough that (3.34) is satisfied, M maps X into itself.

We also wish to ensure that M is a contraction map. Let $\tau_1, \tau_2 \in X$, we have

$$\begin{aligned} \|M(\tau_1) - M(\tau_2)\| \leq& 2a^{-2} \int_0^f \phi'(\theta) \frac{\int_0^\theta |\tau_1(s) - \tau_2(s)| \mathrm{d}s}{\left\{\int_0^\theta [1 - \tau_1(s)]\mathrm{d}s + \frac{2\gamma}{a}\right\} \left\{\int_0^\theta [1 - \tau_2(s)]\mathrm{d}s + \frac{2\gamma}{a}\right\}} \\ \leq& 8a^{-2} \int_0^\mu \phi'(\theta) \frac{\theta}{\left(\theta + \frac{4\gamma}{a}\right)^2} \mathrm{d}\theta \, \|\tau_1 - \tau_2\| \\ \leq& 8a^{-2} \int_0^\mu \frac{\phi'(\theta)}{\theta + \frac{4\gamma}{a}} \mathrm{d}\theta \, \|\tau_1 - \tau_2\|, \end{aligned}$$

and it follows that M is a contraction map if

$$8a^{-2} \int_0^\mu \frac{\phi'(\theta)}{\theta + \frac{4\gamma}{a}} \mathrm{d}\theta < 1.$$

This constitutes our second restriction on μ, it clearly implies the first one, (3.34). The result now follows from the standard fixed-point principle [12].□

For any $a > 0$, the unique positive solution $f(\eta)$, defined in a left-neighbourhood of $\eta = a$, which satisfies the boundary conditions (3.28), may be uniquely continued backward as a function of η. By Lemma 3.4, it will increase monotonically as η decreases. There are then two possibilities, either the solution can be continued back to $\eta = 0$, or else we have $f(\eta) \to \infty$ as η decreases towards some non-negative value. We now show that the solution can indeed be continued back to $\eta = 0$.

Lemma 3.16. *For any given a, γ, the unique local solution of equation (3.26) in Lemma 3.14 can be continued back to $\eta = 0$.*

Proof. Suppose $0 \le a_1 < a$ and $f(\eta) \to \infty$ as $\eta \to a_1$. If there exist $a_2 \in (a_1, a)$ is such that $f(a_2) > \frac{2\gamma}{a}$, then we have from (1.3) that

$$\int_{f(a_2)}^{\infty} \frac{\phi'(f)}{f + \frac{2\gamma}{a}} \mathrm{d}f > \frac{1}{2} \int_{f(a_2)}^{\infty} \frac{\phi'(f)}{f} \mathrm{d}f = \infty. \tag{3.35}$$

But the boundedness of the integral from (3.29), together with (3.35), implies the boundedness of $f(a_1)$.

Now consider $a_1 \le \eta \le a - \delta$ for $\delta > 0$. Integrating (3.26) from η to $a - \delta$ yields

$$-f'(\eta) = \frac{1}{\phi'(f(\eta))} \left(\gamma - \frac{1}{2}(a - \delta) f(a - \delta) + \eta f(\eta) + \int_{\eta}^{a-\delta} f(s) \mathrm{d}s \right),$$

which implies for some constant C that $-f'(\eta) \le C$ for $\eta \le a - \delta$. It follows from [6, Theorem 1.186] that the solution can be continued back to $\eta = 0$.$\square$

Now recall $b(a, \gamma) = \lim_{\eta \to 0} f(\eta; a, \gamma)$, where $\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta)) f'(\eta)$ with $\gamma > 0$. Next we discuss the properties of $b(a, \gamma)$.

3.2.2 Properties of $b(a, \gamma)$

The following discussions on $b(a, \gamma)$ are used in proving existence of self-similar solution by shooting from $\eta = a$ with a given choice of γ, the derivative of $\phi(f)$, back to $\lim_{\eta \to 0} f(\eta; a, \gamma)$.

Lemma 3.17. *$b(a, \gamma)$ has the following properties with fixed a:*

(i) *$b(a, \gamma)$ is strictly monotonically increasing in γ;*

(ii) *$b(a, \gamma)$ is a continuous function of γ and the Lipschitz constant is uniform in $\gamma \in [\gamma_0, \gamma_3]$, where $0 \leq \gamma_0 \leq \gamma_3$;*

(iii) $\lim_{\gamma \to \infty} b(a, \gamma) = \infty$.

Proof. (i) Denote $f_{\gamma_i} = f(\eta; a, \gamma_i)$. Let f_{γ_1} and f_{γ_2} be positive solutions satisfying (3.26), (3.28) corresponding to $\gamma = \gamma_1, \gamma = \gamma_2$. Suppose $b(a, \gamma)$ is not strictly monotonically increasing in γ. Then it is possible to find $\gamma_1 > \gamma_2$ such that $b(a, \gamma_1) \leq b(a, \gamma_2)$ and $\eta_0 \in [0, a)$ such that $f_{\gamma_1}(\eta_0) = f_{\gamma_2}(\eta_0)$ and $f_{\gamma_1} > f_{\gamma_2}$ on (η_0, a), we denote $\bar{f} := f_{\gamma_1}(\eta_0) = f_{\gamma_2}(\eta_0)$.

Integrating the equation (3.26) for f_{γ_1} and f_{γ_2} from η_0 to a and obtain,

$$\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{\eta_0}^{a} f_{\gamma_1}(s)\mathrm{d}s = -\gamma_1 - \phi'(\bar{f}) f'_{\gamma_1}(\eta_0), \tag{3.36}$$

$$\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{\eta_0}^{a} f_{\gamma_2}(s)\mathrm{d}s = -\gamma_2 - \phi'(\bar{f}) f'_{\gamma_2}(\eta_0). \tag{3.37}$$

Subtract (3.37) from (3.36) gives

$$\frac{1}{2}\int_{\eta_0}^{a} (f_{\gamma_1}(s) - f_{\gamma_2}(s))\mathrm{d}s = (\gamma_2 - \gamma_1) + \phi'(\bar{f})[f'_{\gamma_2}(\eta_0) - f'_{\gamma_1}(\eta_0)].$$

Since $f_{\gamma_1} > f_{\gamma_2}$ on (η_0, a), the left-hand side is positive. The right-hand side is negative because $f'_{a_2}(\eta_0) \leq f'_{a_1}(\eta_0)$ at η_0 and $\gamma_2 < \gamma_1$. We therefore have a contradiction. The function $b(a,\gamma)$ must therefore be strictly monotonically increasing in γ.

(ii) Let $0 < \gamma_0 \le \gamma_1 < \gamma_2 \le \gamma_3$. Recall the function $\tau(f)$ from Lemma 3.14 and set $\tau(f) = \tau(f;\gamma_i) = \tau_i$, where $i = 1, 2$. Then

$$\begin{aligned}
&|\tau(f;\gamma_1) - \tau(f;\gamma_2)| \\
=&2a^{-2}\left|\int_0^f \frac{\phi'(\theta)}{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s + \frac{2\gamma_1}{a}}\mathrm{d}\theta - \int_0^f \frac{\phi'(\theta)}{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s + \frac{2\gamma_2}{a}}\mathrm{d}\theta\right| \\
=&2a^{-2}\left|\int_0^f \frac{\phi'(\theta)\left\{\int_0^\theta[\tau_1(s)-\tau_2(s)]\mathrm{d}s + \frac{2\gamma_2}{a} - \frac{2\gamma_1}{a}\right\}}{\left\{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s + \frac{2\gamma_1}{a}\right\}\left\{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s + \frac{2\gamma_2}{a}\right\}}\mathrm{d}\theta\right|.
\end{aligned}$$

Consider the function

$$L(\theta;\gamma) = \left(\theta + \frac{2\gamma}{a}\right)^{-1}\left\{\int_0^\theta[1-\tau(s;\gamma)]\mathrm{d}s + \frac{2\gamma}{a}\right\}, \quad 0 < \theta \le b(a,\gamma).$$

$L(\theta;\gamma)$ is a monotonically decreasing function of θ since

$$\begin{aligned}
\frac{\partial L}{\partial \theta} =& \frac{\left(\theta + \frac{2\gamma}{a}\right)[1-\tau(\theta)] - \int_0^\theta[1-\tau(s)]\mathrm{d}s - \frac{2\gamma}{a}}{\left(\theta + \frac{2\gamma}{a}\right)^2} \\
=& \frac{\int_0^\theta[\tau(s)-\tau(\theta)]\mathrm{d}s - \frac{2\gamma}{a}(1+\tau(\theta))}{\left(\theta + \frac{2\gamma}{a}\right)^2} < 0,
\end{aligned}$$

and $L \to 1$ as $\theta \to 0$. Therefore, when $0 < \theta \le b(a,\gamma)$

$$L[b(a,\gamma);\gamma] \le L(\theta;\gamma) \le 1.$$

We can now write

$$|\tau(f;\gamma_1) - \tau(f;\gamma_2)| \le A(\gamma_2 - \gamma_1) + B\int_0^f \frac{\phi'(\theta)}{\theta + \frac{2\gamma_1}{a}} \max_{0\le s\le\theta}|\tau(s;\gamma_1) - \tau(s;\gamma_2)|\mathrm{d}\theta,$$

where

$$\begin{aligned}
A &= 16a^{-1}\gamma_0^{-2}\phi(b(a,\gamma_3))\left\{L[b(a,\gamma_1);\gamma_1]\right\}^{-1}\left\{L[b(a,\gamma_2);\gamma_2]\right\}^{-1}, \\
B &= 2a^{-2}\left\{L[b(a,\gamma_1);a]\right\}^{-1}\left\{L[b(a,\gamma_2);\gamma_2]\right\}^{-1},
\end{aligned}$$

and if we set $\omega(f) = \max_{0\le\theta\le f} |\tau(\theta;\gamma_1) - \tau(\theta;\gamma_2)|$, then

$$\omega(f) \le A(\gamma_2 - \gamma_1) + B\int_0^f \frac{\phi'(\theta)}{\theta + \frac{2\gamma_1}{a}}\omega(\theta)\mathrm{d}\theta.$$

Define the function

$$M(\gamma) = L[b(a,\gamma);\gamma] = [a\,b(a,\gamma) + 2\gamma]^{-1}\left[\int_0^a f(\eta;\gamma)\mathrm{d}\eta + 2\gamma\right].$$

It was shown in (ii) that, since $\gamma_i \ge \gamma_0 (i = 1, 2)$,

$$f(\eta;\gamma_i) \ge f(\eta;\gamma_0) \quad \text{on } [0,a).$$

Since $f(\eta;\gamma_i) > 0$ it follows that

$$M(\gamma_i) \ge [a\,b(a,\gamma_i) + 2\gamma_i]^{-1}\left[\int_0^a f(\eta;\gamma_0)\mathrm{d}\eta + 2\gamma_0\right].$$

Moreover, $\gamma_i \le \gamma_3$ and hence, in view of (ii), $b(a,\gamma_i) < b(a,\gamma_3)$. Therefore

$$M(\gamma_i) \ge [a\,b(a,\gamma_3) + 2\gamma_3]^{-1}\left[\int_0^a f(\eta;\gamma_0)\mathrm{d}\eta + 2\gamma_0\right].$$

Thus it can be seen that the constants A and B are uniformly bounded for $\gamma \in [\gamma_0, \gamma_3]$.

It now follows from Gronwall's Lemma (see [20, p.24]) and the fact that $f \le b(a,\gamma_3)$, that $\tau(f;\gamma)$ satisfies a Lipschitz condition in γ which is uniform with respect to $f \in [0, b(a,\gamma_3)]$ and $\gamma \in [\gamma_0, \gamma_3]$.

From this, and the observation that τ is continuously differentiable on $(0, 1]$ with

$$\frac{\partial\tau}{\partial f} = 2a^{-2}\frac{\phi'(f)}{f + \frac{2\gamma}{a}}[L(f;\gamma)]^{-1} \ge 2a^{-2}\frac{\phi'(f)}{f + \frac{2\gamma}{a}},$$

we can write

$$
\begin{aligned}
|\tau(b(a,\gamma_1);\gamma_2)-\tau(b(a,\gamma_2);\gamma_2)| =& \int_{b(a,\gamma_1)}^{b(a,\gamma_2)} \frac{\partial\tau}{\partial f}(f,\gamma_2)\mathrm{d}f \\
\geq& 2a^{-2}\frac{\phi'(f^*)}{f^*+\frac{2\gamma_2}{a}}[b(a,\gamma_2)-b(a,\gamma_1)],
\end{aligned}
$$

by the Mean Value Theorem, for some $f^* \in (b(a,\gamma_1), b(a,\gamma_2))$. Now we consider

$$
\begin{aligned}
&|\tau(b(a,\gamma_1);\gamma_2)-\tau(b(a,\gamma_2);\gamma_2)| - |\tau(b(a,\gamma_1);\gamma_1)-\tau(b(a,\gamma_1);\gamma_2)| \\
\leq& |\tau(b(a,\gamma_1);\gamma_1)-\tau(b(a,\gamma_2);\gamma_2)| = 0.
\end{aligned}
$$

Then we have

$$
|\tau(b(a,\gamma_1);\gamma_2)-\tau(b(a,\gamma_2);\gamma_2)| \leq |\tau(b(a,\gamma_1);\gamma_1)-\tau(b(a,\gamma_1);\gamma_2)| \leq K|\gamma_1-\gamma_2|,
$$

since τ is Lipschitz continuous. Therefore

$$
2a^{-2}\frac{\phi'(f^*)}{f^*+\frac{2\gamma_2}{a}}[b(a,\gamma_2)-b(a,\gamma_1)] \leq K|\gamma_1-\gamma_2|.
$$

We may conclude that the function $b(a,\gamma)$ Lipschitz continuous in γ and the Lipschitz constant is uniform in $\gamma \in [\gamma_0, \gamma_3]$.

(iii) Integrating (3.26) from η to a yields

$$
-\phi'(f(\eta))f'(\eta) = \gamma - \frac{1}{2}\int_{\eta}^{a} sf'(s)\mathrm{d}s \geq \gamma.
$$

Then we integrate from η to a and obtain

$$
\int_0^{f(\eta)} \phi'(f)\mathrm{d}f \geq \gamma(a-\eta),
$$

letting $\eta \to 0$ gives

$$
\int_0^{b(a,\gamma)} \frac{\phi'(f)}{f}\mathrm{d}f \geq a\gamma.
$$

As $\phi'(f)$ is continuous on $[0, \infty)$ and $\phi'(0) = 0$ then we have $b(a, \gamma) \to \infty$ as $\gamma \to \infty$. $\square$

Next, we discuss properties of $b(a, \gamma)$ with fixed γ.

Lemma 3.18. *$b(a, \gamma)$ has the following properties with fixed γ:*

(i) *$b(a, \gamma)$ is strictly monotonically increasing in a;*

(ii) $\lim_{a \to 0} b(a, \gamma) = 0$;

(iii) *$b(a, \gamma)$ is Lipschitz continuous in a and the Lipschitz constant is uniform in $a \in (a_0, a_3)$ and $\gamma \in (\gamma_0, \gamma_3)$, where $0 \leq a_0 \leq a_3$, $0 \leq \gamma_0 \leq \gamma_3$;*

(iv) $\lim_{a \to \infty} b(a, \gamma) = \infty$.

Proof. (i) Denote $f_{a_i} = f(\eta; a_i, \gamma)$. Let f_{a_1} and f_{a_2} be positive solutions satisfying (3.26), (3.28) and corresponding to $a = a_1, a = a_2$. Suppose $b(a, \gamma)$ is not strictly monotonically increasing in a. Then it is possible to find $0 < a_1 < a_2$ such that $b(a_2, \gamma) \leq b(a_1, \gamma)$ and $\eta_0 \in (0, a_1)$ such that $f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$ and $f_{a_1} < f_{a_2}$ on (η_0, a_1), we denote $\bar{f} := f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$.

Integrating the equation for f_{a_1} from η_0 to a_1 and the equation for f_{a_2} from η_0 to a_2 yield,

$$\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{\eta_0}^{a_1} f_{a_1}(s)\mathrm{d}s = -\gamma - \phi'(\bar{f}) f'_{a_1}(\eta_0), \tag{3.38}$$

$$\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{\eta_0}^{a_2} f_{a_2}(s)\mathrm{d}s = -\gamma - \phi'(\bar{f}) f'_{a_2}(\eta_0). \tag{3.39}$$

Subtracting (3.38) from (3.39) gives

$$\frac{1}{2}\int_{\eta_0}^{a_1} (f_{a_2}(s) - f_{a_1}(s))\mathrm{d}s + \frac{1}{2}\int_{a_1}^{a_2} f_{a_2}(s)\mathrm{d}s + \phi'(\bar{f})[f'_{a_2}(\eta_0) - f'_{a_1}(\eta_0)] = 0.$$

Since $f_{a_1} < f_{a_2}$ on (η_0, a_1) and $f_{a_2} > 0$ on (a_1, a_2), the first and second term are positive. The third term is non-negative because $f'_{a_2}(\eta_0) \geq f'_{a_1}(\eta_0)$ at η_0. We therefore have a contradiction. The function $b(a,\gamma)$ must therefore be monotonically increasing in a.

(ii) Let $a < 1$, denote $N = \phi'(b(1,\gamma))$, we have $\phi'(f) \leq N$ by (i). Then we get directly from (3.26) that

$$-\frac{\eta}{2N}[\phi(f(\eta))]' \leq [\phi(f(\eta))]'' \quad \text{for } 0 < \eta < a,$$

which implies that

$$\left\{ e^{\frac{\eta^2}{4N}} [\phi(f(\eta))]' \right\}' \geq 0,$$

and integrating from 0 to η then yields

$$[\phi(f(\eta))]' \geq A e^{\frac{-\eta^2}{4N}},$$

where $A = \phi'(b(a,\gamma))f'(0) < 0$. Integrating from η to a we get

$$\phi(f(\eta)) \leq -A \int_\eta^a e^{\frac{-s^2}{4N}} \mathrm{d}s.$$

Now we integrate the equation (3.26) from 0 to a and obtain

$$\frac{1}{2}\int_0^a f(s)\mathrm{d}s = \gamma - \phi'(b(a,\gamma))f'(0).$$

Then $-A = \gamma + \frac{1}{2}\int_0^a f(s)\mathrm{d}s$ and we have $-A \to \gamma$ is bounded as $a \to 0$.

Therefore $\lim\limits_{\eta \to 0} \phi(f(\eta)) \leq -A \int_0^a e^{\frac{-s^2}{4N}} \mathrm{d}s \to 0$ as $a \to 0$ since $e^{\frac{-s^2}{4N}}$ is bounded, which implies that $\lim\limits_{a\to 0}\lim\limits_{\eta \to 0} f(\eta) = \lim\limits_{a \to 0} b(a,\gamma) = 0$.

(iii) Let $0 < a_0 \leq a_1 < a_2 \leq a_3$. Recall the function $\tau(f)$ from Lemma 3.14 and set $\tau(f) = \tau(f; a_i) = \tau_i$, where $i = 1, 2$. Then

$$
\begin{aligned}
&|\tau(f;a_1)-\tau(f;a_2)| \\
=&\left|2a_1^{-2}\int_0^f \frac{\phi'(\theta)}{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}}\mathrm{d}\theta-2a_2^{-2}\int_0^f \frac{\phi'(\theta)}{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s-\frac{2\gamma}{a_2}}\mathrm{d}\theta\right|, \\
=&\left|2\int_0^f \frac{\phi'(\theta)\left\{a_2^2\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+2a_2\gamma-a_1^2\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+2a_1\gamma\right\}}{a_1^2a_2^2\left\{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}\right\}\left\{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+\frac{2\gamma}{a_2}\right\}}\right|, \\
=&2a_2^{-2}\int_0^f \frac{\phi'(\theta)\int_0^\theta|\tau_1(s)-\tau_2(s)|\mathrm{d}s}{\left\{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}\right\}\left\{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+\frac{2\gamma}{a_2}\right\}}\mathrm{d}\theta \\
&+\frac{2(a_2^2-a_1^2)}{a_2^2a_1^2}\int_0^f \frac{\phi'(\theta)\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+\frac{2\gamma}{a_1+a_2}\mathrm{d}\theta}{\left\{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}\right\}\left\{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+\frac{2\gamma}{a_2}\right\}} \\
\leq&2a_2^{-2}\int_0^f \frac{\phi'(\theta)\int_0^\theta|\tau_1(s)-\tau_2(s)|\mathrm{d}s}{\left\{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}\right\}\left\{\int_0^\theta[1-\tau_2(s)]\mathrm{d}s+\frac{2\gamma}{a_2}\right\}}\mathrm{d}\theta \\
&+\frac{2(a_2^2-a_1^2)}{a_2^2a_1^2}\int_0^f \frac{\phi'(\theta)}{\int_0^\theta[1-\tau_1(s)]\mathrm{d}s+\frac{2\gamma}{a_1}}\mathrm{d}\theta.
\end{aligned}
$$

Let $0<\gamma_0\leq\gamma_1<\gamma_2\leq\gamma_3$ and consider the function

$$
L(\theta;a,\gamma)=\left(\theta+\frac{2\gamma}{a}\right)^{-1}\left\{\int_0^\theta[1-\tau(s;a)]\mathrm{d}s+\frac{2\gamma}{a}\right\},\quad 0<\theta\leq b(a,\gamma).
$$

$L(\theta;a)$ is clearly a monotonically decreasing function of θ, and $L\to 1$ as $\theta\to 0$. Therefore

$$
L[b(a,\gamma);a]\leq L(\theta;a)\leq 1\quad\text{for } 0<\theta\leq b(a,\gamma).
$$

It then follows that

$$
|\tau(f;a_1)-\tau(f;a_2)|\leq A(a_2-a_1)+B\int_0^f\frac{\phi'(\theta)}{\theta+\frac{2\gamma}{a}}\max_{0\leq s\leq\theta}|\tau(s;a_1)-\tau(s;a_2)|\mathrm{d}\theta,
$$

where

$$
\begin{aligned}
A &= 2\frac{a_1+a_2}{a_1^2a_2^2}\int_0^{b(a_1,\gamma)}\frac{\phi'(\theta)}{\theta+\frac{2\gamma_0}{a_1}}\mathrm{d}\theta\left\{L[b(a_1,\gamma);a_1]\right\}^{-1}\\
&\leq 2\frac{a_1+a_2}{a_1^2a_2^2}\int_0^{b(a_1,\gamma_3)}\frac{\phi'(\theta)}{\theta+\frac{2\gamma_0}{a_1}}\mathrm{d}\theta\left\{L[b(a_1,\gamma);a_1]\right\}^{-1},\\
B &= 2a_2^{-2}\left\{L[b(a_1,\gamma);a_1]\right\}^{-1}\left\{L[b(a_2,\gamma);a_2]\right\}^{-1}.
\end{aligned}
$$

If we set $\omega(f) = \max_{0\leq\theta\leq f}|\tau(\theta;a_1)-\tau(\theta;a_2)|$, we then have

$$
\omega(f) \leq A(a_2-a_1) + B\int_0^f \frac{\phi'(\theta)}{\theta+\frac{2\gamma}{a}}\omega(\theta)\mathrm{d}\theta.
$$

Define the function

$$
M(a,\gamma) := L[b(a,\gamma);a] = [a\,b(a,\gamma)+2\gamma]^{-1}\left[\int_0^a f(\eta;a,\gamma)\mathrm{d}\eta + 2\gamma\right].
$$

It was shown in the proof of (i) that, since $a_i \geq a_0 (i=1,2)$, $f(\eta;a_i,\gamma) \geq f(\eta;a_0,\gamma)$ on $[0,a_0)$. Since $f(\eta;a_i,\gamma) > 0$ on $[a_0,a_1)$ it follows that

$$
M(a_i,\gamma) \geq [a_ib(a_i,\gamma)+2\gamma]^{-1}\left[\int_0^{a_0} f(\eta;a_0,\gamma)\mathrm{d}\eta + 2\gamma\right].
$$

By Lemma 3.17 (i), since $\gamma > \gamma_0$, $f(\eta;a,\gamma) \geq f(\eta;a,\gamma_0)$ on $(0,a_0)$, so that

$$
M(a_i,\gamma_i) \geq [a_ib(a_i,\gamma)+2\gamma]^{-1}\left[\int_0^{a_0} f(\eta;a_0,\gamma_0)\mathrm{d}\eta + 2\gamma_0\right].
$$

Moreover, $a_i \leq a_3$, $\gamma \leq \gamma_3$ and hence, $b(a_i,\gamma) < b(a_3,\gamma_3)$. Therefore

$$
M(a_i,\gamma_i) \geq [a_3b(a_3,\gamma_3)+2\gamma_3]^{-1}\left[\int_0^{a_0} f(\eta;a_0,\gamma_0)\mathrm{d}\eta + 2\gamma_0\right].
$$

Thus it can be seen that the constant A and B are uniformly bounded on the interval $[a_0,a_3]$ and $[\gamma_0,\gamma_3]$.

It now follows from Gronwall's Lemma [20, p24] and the fact $f \leq b(a_3,\gamma)$ that $\tau(f;a)$ satisfies a Lipschitz condition in a which is uniform with respect to $f \in [0,b(a_3,\gamma)]$ and $a \in [a_0,a_3]$.

From this, and the observation that τ is continuously differentiable on $(0,1]$ with

$$\frac{\partial \tau}{\partial f} = 2a^{-2}\frac{\phi'(f)}{f+\frac{2\gamma}{a}}[L(f;a)]^{-1} \geq 2a^{-2}\frac{\phi'(f)}{f+\frac{2\gamma}{a}},$$

since $[L(f;a)] \leq 1$.

We can write

$$\begin{aligned}|\tau(b(a_1,\gamma);a_2)-\tau(b(a_2,\gamma);a_2)| = &\left|\int_{b(a_1,\gamma)}^{b(a_2,\gamma)}\frac{\partial \tau}{\partial f}(f,a_2)\mathrm{d}f\right| \\ \geq &2a_2^{-2}\frac{\phi'(f^*)}{f^*+\frac{2\gamma_3}{a_2}}|b(a_2,\gamma)-b(a_1,\gamma)|,\end{aligned}$$

by the Mean Value Theorem, for some $f^* \in (b(a_1,\gamma), b(a_2,\gamma))$ and $f^* \geq b(a_0,\gamma) > 0$.

We may conclude by a similar argument to that in proof of Lemma 3.17 (ii) that the function $b(a,\gamma)$ is Lipschitz continuous in a and the Lipschitz constant is uniform in $a \in (a_0, a_3)$ and $\gamma \in (\gamma_0, \gamma_3)$.

Here we can prove the Lipshichtz constant is uniform in both $a \in (a_0, a_3)$ and $\gamma \in (\gamma_0, \gamma_3)$ since we proved the monotonicity on γ of $b(a,\gamma)$ on Lemma 3.17. This result will be used to prove $b(a,\gamma)$ is a continuous function of both a and γ.

(iv) Integrating (3.26) from η to a yields

$$-\phi'(f(\eta))f'(\eta) = \gamma - \frac{1}{2}\int_{\eta}^{a} sf'(s)\mathrm{d}s,$$

then we have

$$-\phi'(f(\eta))f'(\eta) \geq \gamma + \frac{\eta}{2}f(\eta) \geq \frac{\eta}{2}f(\eta).$$

For any a_4 with $\eta < a_4 < a$ we obtain

$$\int_{f(a_4)}^{f(\eta)}\frac{\phi'(f)}{f}\mathrm{d}f \geq \frac{1}{4}(a_4^2-\eta^2),$$

letting $a_4 \to a$ and $\eta \to 0$,

$$\int_0^{b(a,\gamma)} \frac{\phi'(f)}{f} \mathrm{d}f \geq \frac{1}{4}a^2.$$

As $\phi'(f)$ is continuous on $[0,\infty)$ and $\phi'(0) = 0$ then we have $b(a,\gamma) \to \infty$ as $a \to \infty$. $\square$

Now we prove that $b(a,\gamma)$ is a continuous function of both a and γ by using Lemma 3.17 (ii) and Lemma 3.18 (iii).

Lemma 3.19. *$b(a,\gamma)$ is a continuous function of γ and a.*

Proof. Consider

$$|b(a,\gamma) - b(a_0,\gamma_0)| \leq |b(a,\gamma) - b(a_0,\gamma)| + |b(a_0,\gamma) - b(a_0,\gamma_0)|.$$

It was shown in the proof of Lemma 3.17 (ii) that $b(a,\gamma)$ is uniformly continuous in $\gamma \in [\gamma_0,\gamma_3]$, so there exists μ_1 such that $|b(a_0,\gamma) - b(a_0,\gamma_0)| < \frac{\delta}{2}$ if $|\gamma - \gamma_0| < \mu_1$. And by the proof of Lemma 3.18 (iii), there exists μ_2 such that $|b(a,\gamma) - b(a_0,\gamma)| < \frac{\delta}{2}$ if $|a - a_0| < \mu_2$ and $\gamma \in [\gamma_0,\gamma_3]$. Therefore $|b(a,\gamma) - b(a_0,\gamma_0)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ if $|a - a_1| + |\gamma - \gamma_0| \leq \max\{\mu_1,\mu_2\}$. $\square$

By similar arguments to those in Lemmas 3.17, 3.18 and 3.19 and the fact that the particular choice of $\eta = 0$ in $b(a,\gamma) = f(0;a,\gamma)$ plays no special role, letting $\eta_0 \in [0,\infty]$ play the same role as 0, we can obtain the following corollary.

Corollary 3.20. *For each fixed $\eta_0 \in (0,a)$, if f satisfies (3.26) and (3.28), then $f(\eta_0;a,\gamma)$ is a continuous function of a and γ and is monotonically increasing in both a and γ.*

3.2.3 Solution in right-neighbourhood of $\eta = a$

Now we consider f that satisfies the equation

$$-\frac{1}{2}\eta f'(\eta) = \varepsilon[\phi'(-f(\eta))f'(\eta)]', \quad \eta > a. \tag{3.40}$$

At the boundaries we require

$$\lim_{\eta\to\infty} f(\eta) = -V_0, \tag{3.41}$$

$$\lim_{\eta\searrow a} f(\eta) = 0, \quad \lim_{\eta\searrow a} \varepsilon\phi'(-f(\eta))f'(\eta) = -\gamma. \tag{3.42}$$

We next prove the existence of a negative solution of (3.40) in a right-neighbourhood of $\eta = a$, which satisfies the boundary conditions (3.42).

Lemma 3.21. *For given $a > 0$, there exists $\delta > 0$ such that in $(a, a + \delta)$ equation (3.40) has a unique solution which is negative and satisfies the boundary condition (3.42).*

Proof. It is convenient to start by supposing that such a solution exists in a right-neighbourhood of $\eta = a$. Integrating (3.40) from a to η then yields

$$-\frac{1}{2}\int_a^\eta sf'(s)\mathrm{d}s = \gamma + \varepsilon\phi'(-f(\eta))f'(\eta), \tag{3.43}$$

we write (3.43) in the form

$$\frac{1}{f'(\eta)} = \frac{2\varepsilon\phi'(-f)}{-\int_a^\eta sf'(s)\mathrm{d}s - 2\gamma}. \tag{3.44}$$

Since f is monotonic by Lemma 3.4, with non-vanishing derivative, we can treat η as a function of f, writing $\eta = \sigma(f)$. Then (3.44) takes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}f} = \frac{2\varepsilon\phi'(-f)}{\int_f^0 \sigma(s)\mathrm{d}s - 2\gamma},$$

and $\sigma(f)$ is a solution of this integro-differential equation which satisfies the initial condition $\sigma(0) = a$ and is defined and continuous on an interval $[f_0, 0]$

for some $f_0 < 0$, and is continuously differentiable on $(f_0, 0)$. An integration gives

$$\sigma(f) = a - 2\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \sigma(s)\mathrm{d}s - 2\gamma}\mathrm{d}\theta, \tag{3.45}$$

and if we set

$$\tau(f) = \frac{a}{\sigma(f)} = \frac{a}{\eta},$$

then (3.45) becomes

$$\tau(f) = \frac{1}{1 - 2a^{-2}\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma}{a}}\mathrm{d}\theta}. \tag{3.46}$$

If the solution of (3.46) is unique, the corresponding solution of equation (3.40) is also unique.

Lemma 3.22. *There exists a $\mu > 0$ such that (3.46) has a unique continuous solution in $-\mu \le f \le 0$, which is such that $\tau(0) = 1$ and $\tau(f) < 1$ if $-\mu \le f < 0$.*

Proof. With μ to be chosen later, we denoted by X the set of continuous functions $\tau(f)$ defined in $[-\mu, 0]$, satisfying $\frac{1}{2} \le \tau(f) \le 1$. We denote by $\|\,.\,\|$ the supremum norm on X. Then X is a complete metric space. On X we introduce the map

$$\begin{aligned} M(\tau)(f) =& \frac{1}{1 - 2a^{-2}\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma}{a}}\mathrm{d}\theta} \\ \ge& \frac{1}{1 + 2a^{-2}\int_{-\mu}^0 \frac{\varepsilon\phi'(-\theta)}{\theta + \frac{2\gamma}{a}}\mathrm{d}\theta}. \end{aligned}$$

It is clear that $M(\tau)(f) \le 1$ is well-defined, continuous. Moreover, $M(\tau) \ge \frac{1}{2}$ if

$$\frac{1}{1 + 2a^{-2}\int_{-\mu}^0 \frac{\varepsilon\phi'(-\theta)}{\theta + \frac{2\gamma}{a}}\mathrm{d}\theta} \ge \frac{1}{2},$$

which gives

$$2a^{-2}\int_{-\mu}^{0}\frac{\varepsilon\phi'(-\theta)}{\theta+\frac{2\gamma}{a}}\mathrm{d}\theta \leq 1. \tag{3.47}$$

Therefore, if μ is chosen so small that (3.47) is satisfied, M maps X into itself.

We also wish to ensure that M is a contraction map. Let $\tau_1, \tau_2 \in X$ and choose $\mu \leq \frac{\gamma}{a}$, we have

$$\begin{aligned}
\|M(\tau_1)-M(\tau_2)\| =& 2a^{-2}\left\|\frac{\int_f^0 \varepsilon\phi'(-\theta)\frac{\int_\theta^0 \frac{1}{\tau_2(s)}-\frac{1}{\tau_1(s)}\mathrm{d}s}{(\int_\theta^0\frac{1}{\tau_1(s)}\mathrm{d}s-\frac{2\gamma}{a})(\int_\theta^0\frac{1}{\tau_2(s)}\mathrm{d}s-\frac{2\gamma}{a})}}{(1-2a^{-2}\int_f^0\frac{\varepsilon\phi'(-\theta)}{\int_\theta^0\frac{1}{\tau_1(s)}\mathrm{d}s-\frac{2\gamma}{a}})(1-2a^{-2}\int_f^0\frac{\varepsilon\phi'(-\theta)}{\int_\theta^0\frac{1}{\tau_2(s)}\mathrm{d}s-\frac{2\gamma}{a}})}\right\| \\
\leq& 2a^{-2}\left\|\int_{-\mu}^{0}\varepsilon\phi'(-\theta)\frac{\int_\theta^0\frac{\tau_1(s)-\tau_2(s)}{\tau_1(s)\tau_2(s)}\mathrm{d}s}{(\theta+\frac{2\gamma}{a})^2}\mathrm{d}\theta\right\| \\
\leq& 8a^{-2}\int_{-\mu}^{0}\frac{-\theta\varepsilon\phi'(-\theta)}{(\theta+\frac{2\gamma}{a})^2}\mathrm{d}\theta\|\tau_1-\tau_2\| \\
\leq& 8a^{-2}\int_{-\mu}^{0}\frac{\varepsilon\phi'(-\theta)}{\theta+\frac{2\gamma}{a}}\mathrm{d}\theta\|\tau_1-\tau_2\|.
\end{aligned}$$

It follows that M is a contraction map if

$$8a^{-2}\int_{-\mu}^{0}\frac{\varepsilon\phi'(-\theta)}{\theta+\frac{2\gamma}{a}}\mathrm{d}\theta < 1.$$

This constitutes our third restriction on μ, it clearly implies the first one. The result now follows from a standard fixed-point principle [12]. □

For any $a > 0$, the unique negative solution $f(\eta)$ defined in a right-neighbourhood of $\eta = a$, which satisfies the boundary conditions (3.42), may be uniquely continued forward as a function of η. By the monotonicity, it will decrease monotonically as η increases. There are then two possibilities. Either the solution can be continued forward to $\eta \to \infty$, or else we have $f(\eta) \to -\infty$ as η increases towards some positive value. We now show that the solution can be continued forward to $\eta \to \infty$.

Lemma 3.23. *For given a, γ, the unique local solution in Lemma 3.21 can be continued forward to $\eta \to \infty$.*

Proof. We have from (3.40) that

$$-f'(\eta) \leq \frac{2\varepsilon}{a}[\phi'(-f)f']'. \tag{3.48}$$

Integrating (3.48) from $2a$ to η yields

$$-\varepsilon\phi'(-f(2a))f'(2a) - \frac{a}{2}f(2a) \geq -\varepsilon\phi'(-f(\eta))f'(\eta) - \frac{a}{2}f(\eta),$$

then we know that $-\varepsilon\phi'(-f(\eta))f'(\eta) - \frac{a}{2}f(\eta)$ is bounded above by some positive constant C, so f is bounded. The boundedness of $-f'(\eta)$ for $\eta > a + \frac{\delta}{2}$ follows similarly to (3.18) for $\delta > 0$, so it follows from [6, Theorem 1.186] that the solution of (3.40) can be continuous forward to $\eta \to \infty$. $\square$

Now recall $d(a, \gamma) = \lim_{\eta\to\infty} f(\eta; a, \gamma)$. Next we discuss the properties of $d(a, \gamma)$.

3.2.4 Properties of $d(a, \gamma)$

The following discussions on $d(a, \gamma)$ are used in proving existence of self-similar solution by shooting from $\eta = a$ with γ, the derivatives of $\phi(f)$, to $\lim_{\eta\to\infty} f(\eta; a, \gamma)$.

Lemma 3.24. *$d(a, \gamma)$ has the following properties with fixed a:*

(i) *$d(a, \gamma)$ is strictly monotonically decreasing in γ;*

(ii) *$\lim_{\gamma\to\infty} d(a, \gamma) = -\infty$;*

(iii) *$\lim_{\gamma\to 0} d(a, \gamma) = 0$.*

Proof. (i) Denote $f_{\gamma_i} = f(\eta; a, \gamma_i)$. Let f_{γ_1} and f_{γ_2} be positive solutions satisfy (3.40) and (3.42) corresponding to $\gamma = \gamma_1, \gamma = \gamma_2$. Suppose now $d(a,\gamma)$ is not strictly monotonically decreasing in γ. Then it is possible to find $\gamma_1 \geq \gamma_2$ such that $d(a,\gamma_1) > d(a,\gamma_2)$ and $\eta_0 \in (a,\infty)$ such that $f_{\gamma_1}(\eta_0) = f_{\gamma_2}(\eta_0)$ and $f_{\gamma_1} < f_{\gamma_2}$ on (a,η_0), we denote by $\bar{f} := f_{\gamma_1}(\eta_0) = f_{\gamma_2}(\eta_0)$.

Integrating (3.40) for f_{γ_1} and f_{γ_2} from a to η_0 yields,

$$-\frac{1}{2}\eta_0\bar{f} + \frac{1}{2}\int_a^{\eta_0} f_{\gamma_1}(\eta)\mathrm{d}\eta = \phi'(-\bar{f})f'_{\gamma_1}(\eta_0) + \gamma_1, \tag{3.49}$$

$$-\frac{1}{2}\eta_0\bar{f} + \frac{1}{2}\int_a^{\eta_0} f_{\gamma_2}(\eta)\mathrm{d}\eta = \phi'(-\bar{f})f'_{\gamma_2}(\eta_0) + \gamma_2. \tag{3.50}$$

If we subtract (3.50) from (3.49), then

$$\frac{1}{2}\int_a^{\eta_0}[f_{\gamma_1}(\eta) - f_{\gamma_2}(\eta)]\mathrm{d}\eta = \phi'(-\bar{f})[f'_{\gamma_1}(\eta_0) - f'_{\gamma_2}(\eta_0)] + (\gamma_1 - \gamma_2).$$

Since $f_{\gamma_1} < f_{\gamma_2}$ on (a,η_0), the left-hand side is negative. The right-hand side is positive because $f'_{\gamma_1}(\eta_0) \geq f'_{\gamma_2}(\eta_0)$ and $\gamma_1 > \gamma_2$. We therefore have a contradiction. The function $d(a,\gamma)$ must therefore be strictly monotonically decreasing in γ.

(ii) Suppose $d(a,\gamma)$ does not satisfy $\lim\limits_{\gamma\to\infty} d(a,\gamma) = -\infty$. Then there exists $M > 0$ such that $d(a,\gamma) \geq -M$ for all γ, which implies $|f(\eta_0)| \leq M$ for each fixed $\eta_0 > a$.

Integrating (3.40) from a to η_0 gives

$$\gamma = -\frac{1}{2}\eta_0 f(\eta_0) + \frac{1}{2}\int_a^{\eta_0} f(s)\mathrm{d}s - \varepsilon\phi'(-f(\eta_0))f'(\eta_0).$$

Since $\displaystyle\int_a^{\eta_0} f(s)\mathrm{d}s$ is negative, using the upper bound of $|f|$, we get

$$-\varepsilon\phi'(M)f'(\eta_0) > \gamma - \frac{M\eta_0}{2} - \frac{1}{2}\int_a^{\eta_0} f(s)\mathrm{d}s > \gamma - \frac{M\eta_0}{2}.$$

By (3.18) we know that for all $\eta_0 > a+1$

$$\frac{4\gamma\phi'(M)}{2\eta_0 - 1} > -\phi'(M)f'(\eta_0) > \gamma - \frac{M\eta_0}{2}.$$

If we rewrite as $\dfrac{M\eta_0}{2} > \gamma\left(1 - \dfrac{4\phi'(M)}{2\eta_0 - 1}\right)$ and choosing and fixing η_0 sufficient large such that $1 - \dfrac{4\phi'(M)}{2\eta_0 - 1} > \dfrac{1}{2}$, we then have $M\eta_0 > \gamma$ for all γ. But this is a contradiction, so if $\gamma \to \infty$, we have $d(a,\gamma) \to -\infty$.

(iii) Integrating (3.40) from a to η yields

$$-\frac{1}{2}\int_a^\eta sf'(s)\mathrm{d}s = \varepsilon\phi'(-f(\eta))f'(\eta) + \gamma,$$

letting $\eta \to \infty$, together with $\lim_{\eta\to\infty} f'(\eta) = 0$ by (3.17), we get

$$-\frac{a}{2}\int_a^\infty f'(s)\mathrm{d}s \leq \gamma,$$

which implies

$$-\frac{a}{2}d(a,\gamma) \leq \gamma.$$

Then the result follows from

$$-\frac{a}{2}d(a,\gamma) \to 0 \quad \text{as } \gamma \to 0.$$

$\square$

Next, we discuss the properties of $d(a,\gamma)$ for fixed γ.

Lemma 3.25. *$d(a,\gamma)$ has the following properties with fixed γ:*

(i) *$d(a,\gamma)$ is strictly monotonically increasing in a;*

(ii) *$d(a,\gamma)$ is a continuous function of a;*

(iii) $\lim_{a\to\infty} d(a,\gamma) = 0$.

Proof. (i) Denote $f_{a_i} = f(\eta; a_i, \gamma)$. Let f_{a_1} and f_{a_2} be positive solutions of (3.40) and (3.42) corresponding $a = a_1, a = a_2$. Suppose $d(a, \gamma)$ is not strictly monotonically increasing of a. Then it is possible to find $0 < a_1 < a_2$ such that $d(a_1, \gamma) \geq d(a_2, \gamma)$ and $\eta_0 \in (a_2, \infty)$ such that $f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$ and $f_{a_1} < f_{a_2}$ on (a, η_0), we denote $\bar{f} := f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$.

We integrating the equation (3.40) for f_{a_1} from a_1 to η_0 and f_{a_2} from a_2 to η_0, then we obtain

$$-\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{a_1}^{\eta_0} f_{a_1}(s)\mathrm{d}s = \gamma + \varepsilon\phi'(-\bar{f})f'_{a_1}(\eta_0), \tag{3.51}$$

$$-\frac{1}{2}\eta_0 \bar{f} + \frac{1}{2}\int_{a_2}^{\eta_0} f_{a_2}(s)\mathrm{d}s = \gamma + \varepsilon\phi'(-\bar{f})f'_{a_2}(\eta_0). \tag{3.52}$$

Subtracting (3.52) from (3.51) gives

$$\frac{1}{2}\int_{a_2}^{\eta_0} (f_{a_1}(s) - f_{a_2}(s))\mathrm{d}s + \frac{1}{2}\int_{a_1}^{a_2} f_{a_1}(s)\mathrm{d}s = \varepsilon\phi'(-\bar{f})[f'_{a_1}(\eta_0) - f'_{a_2}(\eta_0)].$$

Since $f_{a_1} < f_{a_2}$ on (η_0, a) and $f_{a_1} < 0$ on (a_1, a_2), the left-hand side is negative. The right-hand side is non-negative because $f'_{a_2}(\eta_0) \leq f'_{a_1}(\eta_0)$ at η_0. We therefore have a contradiction. The function $d(a, \gamma)$ must therefore be strictly monotonically increasing of a.

(iii) Integrating (3.40) from a to η we get

$$\varepsilon\phi'(-f(\eta))f'(\eta) \geq -\gamma - \frac{a}{2}\int_a^{\eta} f'(s)\mathrm{d}s = \frac{a}{2}\left(-\frac{2\gamma}{a} - f(\eta)\right),$$

then the result follows from the fact that $f' < 0$, gives $-\frac{2\gamma}{a} - f < 0$. □

Next, we prove f is a continuous function of a and γ respectively. This result will be used to prove $b(a, \gamma)$ is a continuous function of both a and γ.

Lemma 3.26. *For each fixed $\eta^* > a$, if f satisfies (3.40) and (3.42), then*

(i) *$f(\eta^*; a, \gamma)$ is a continuous function of γ for fixed a;*

(ii) $f(\eta^*; a, \gamma)$ *is a continuous function of* a *for fixed* γ.

Proof. First we prove $f(\eta^*; a, \gamma)$ is a continuous function γ.

Let $0 < \gamma_0 \le \gamma_1 < \gamma_2 \le \gamma_3$. Recall the function $\tau(f)$ from the proof of Lemma 3.21 and set $\tau(f) = \tau(f; \gamma_i) = \tau_i$, where $i = 1, 2$. Let $\eta \in (a, \eta_0]$ satisfies $\frac{1}{2} \le \tau(f) < 1$ and

$$\eta_0 f(\eta_0; a, \gamma_3) - 2\gamma_0 < 0, \tag{3.53}$$

for $f(\eta) \in [-\mu, 0]$. Then

$$\begin{aligned}
&|\tau(f;\gamma_1) - \tau(f;\gamma_2)| \\
=&\left| \frac{2a^{-2}\int_f^0 \left(\frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma_1}{a}} - \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma_2}{a}} \right)\mathrm{d}\theta}{\left(1 - 2a^{-2}\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma_1}{a}}\mathrm{d}\theta\right)\left(1 - 2a^{-2}\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma_2}{a}}\mathrm{d}\theta\right)} \right| \\
\le& 2a^{-2}\left| \int_f^0 \frac{\varepsilon\phi'(-\theta)\left(\int_\theta^0 \frac{1}{\tau_2(s)} - \frac{1}{\tau_1(s)}\mathrm{d}s + \frac{2}{a}(\gamma_1 - \gamma_2)\right)}{\left(\int_\theta^0 \frac{1}{\tau_1(s)}\mathrm{d}s - \frac{2\gamma_1}{a}\right)\left(\int_\theta^0 \frac{1}{\tau_2(s)}\mathrm{d}s - \frac{2\gamma_2}{a}\right)}\mathrm{d}\theta \right|,
\end{aligned}$$

since $\tau(f) < 1$ implies $1 - 2a^{-2}\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma}{a}}\mathrm{d}\theta > 1$.

Then we have

$$\begin{aligned}
&|\tau(f;\gamma_1) - \tau(f;\gamma_2)| \\
\le& 2a^{-2}\left| \int_f^0 \frac{\varepsilon\phi'(-\theta)\left(\int_\theta^0 \frac{\tau_1(s) - \tau_2(s)}{\tau_1(s)\tau_2(s)}\mathrm{d}s + \frac{2}{a}(\gamma_1 - \gamma_2)\right)}{\left(\int_\theta^0 \frac{1}{\tau_1(s)}\mathrm{d}s - \frac{2\gamma_1}{a}\right)\left(\int_\theta^0 \frac{1}{\tau_2(s)}\mathrm{d}s - \frac{2\gamma_2}{a}\right)}\mathrm{d}\theta \right| \\
\le& 2a^{-2}\left| \int_f^0 \frac{\varepsilon\phi'(-\theta)\left(4\int_\theta^0 \tau_1(s) - \tau_2(s)\mathrm{d}s + \frac{2}{a}(\gamma_1 - \gamma_2)\right)}{\left(\int_\theta^0 \frac{1}{\tau_1(s)}\mathrm{d}s - \frac{2\gamma_1}{a}\right)\left(\int_\theta^0 \frac{1}{\tau_2(s)}\mathrm{d}s - \frac{2\gamma_2}{a}\right)}\mathrm{d}\theta \right|,
\end{aligned}$$

since $\tau(f) > \frac{1}{2}$.

Consider the function

$$L(\theta;\gamma) = \left(\theta - \frac{2\gamma}{a}\right)^{-1}\left(\int_\theta^0 \frac{1}{\tau(s)}\mathrm{d}s - \frac{2\gamma}{a}\right) > 0, \quad -\mu \le \theta \le 0.$$

L is a monotonically increasing function of θ since

$$\frac{\partial L}{\partial \theta} = \frac{\int_\theta^0 \frac{1}{\tau(\theta)} - \frac{1}{\tau(s)}\mathrm{d}s + \frac{2\gamma}{a}\left(1+\tau(\theta)\right)}{\left(\theta - \frac{2\gamma}{a}\right)^2} > 0,$$

and $L \to 1$ as $\theta \to 0$. Therefore $L(f(\eta_0; a, \gamma); \gamma) \leq L(\theta; \gamma) < 1$ when $-\mu \leq \theta \leq 0$.

We can now write

$$|\tau(f;\gamma_1) - \tau(f;\gamma_2)| \leq A(\gamma_2 - \gamma_1) + B\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\frac{2\gamma_1}{a} - \theta} \max_{f \leq s \leq 0} |\tau(s;\gamma_1) - \tau(s;\gamma_2)|\mathrm{d}\theta,$$

where

$$A = 16a^{-1}\gamma_0^2\varepsilon\phi'(-f(\eta_0; a, \gamma_1))[L(f(\eta_0; a, \gamma_1))]^{-1}[L(f(\eta_0; a, \gamma_2))]^{-1},$$
$$B = 8a^{-2}[L(f(\eta_0; a, \gamma_1))]^{-1}[L(f(\eta_0; a, \gamma_2))]^{-1},$$

and if we set $\omega(f) = \max\limits_{f \leq \theta \leq 0} |\tau(\theta;\gamma_1) - \tau(\theta;\gamma_2)|$, then

$$\omega(f) \leq A(\gamma_2 - \gamma_1) + B\int_f^0 \frac{\varepsilon\phi'(-\theta)}{\frac{2\gamma_1}{a} - \theta}\omega(\theta)\mathrm{d}\theta.$$

Define the function

$$\begin{aligned} M(\gamma) =& L(f(\eta_0; a, \gamma); \gamma) \\ =& (a\, f(\eta_0; a, \gamma); \gamma) - 2\gamma)^{-1}\left(\int_a^{\eta_0} f(s; a, \gamma)\mathrm{d}s - \eta_0 f(\eta_0; a, \gamma) - 2\gamma\right). \end{aligned}$$

It was shown in Lemma 3.24 (i) that, since $\gamma_i \leq \gamma_3$, $f(\eta; \gamma_i) \geq f(\eta; \gamma_3)$ on $(a, \eta_0]$. Since $f(\eta; \gamma_i) < 0$, it follows that

$$M(\gamma_i) \geq (af(\eta_0; a, \gamma_3); \gamma_3) - 2\gamma_3)^{-1}\left(\int_a^{\eta_0} f(s; a, \gamma)\mathrm{d}s - \eta_0 f(\eta_0; a, \gamma_3) - 2\gamma_0\right) > 0.$$

Thus it can be seen that the constants A and B are uniformly bounded for $\gamma \in [\gamma_0, \gamma_3]$.

It now follows from Gronwall's Lemma ([20, p24]) and the fact that $f(\eta; a, \gamma) \geq f(\eta_0; a, \gamma_3)$, $\tau(f; \gamma)$ satisfies a Lipshichtz condition in γ which is uniform with respect to $f \in [f(\eta_0; a, \gamma_3), 0]$ and $\gamma \in [\gamma_0, \gamma_3]$.

The observation that τ is continuously differentiable on $\left[\frac{1}{2}, 1\right)$ with

$$\frac{\partial \tau}{\partial f} = \frac{1}{2} a^{-2} \frac{\varepsilon \phi'(-f)}{\frac{2\gamma}{a} - f} [L(f; \gamma)]^{-1} \geq \frac{1}{2} a^{-2} \frac{\varepsilon \phi'(-f)}{\frac{2\gamma}{a} - f},$$

gives

$$\begin{aligned}|\tau(f(\eta_0; a, \gamma_1); \gamma_1) - \tau(f(\eta_0; a, \gamma_2); \gamma_1)| =& \int_{f(\eta_0; a, \gamma_1)}^{f(\eta_0; a, \gamma_2)} \frac{\partial \tau}{\partial f}(f; \gamma_1) \mathrm{d}f \\ \geq& \frac{1}{2} a^{-2} \frac{\varepsilon \phi'(-f^*)}{\frac{2\gamma}{a} - f^*} [f(\eta_0; a, \gamma_2) - f(\eta_0; a, \gamma_1)],\end{aligned}$$

by the Mean Value Theorem, for some $f^* \in (f(\eta_0; a, \gamma_1), f(\eta_0; a, \gamma_2))$. Now we consider

$$\begin{aligned}&|\tau(f(\eta_0; a, \gamma_1); \gamma_1) - \tau(f(\eta_0; a, \gamma_2); \gamma_1)| - |\tau(f(\eta_0; a, \gamma_1); \gamma_2) - \tau(f(\eta_0; a, \gamma_2); \gamma_2)| \\ \leq& |\tau(f(\eta_0; a, \gamma_1); \gamma_1) - \tau(f(\eta_0; a, \gamma_2); \gamma_2)| = 0.\end{aligned}$$

We may conclude by a similar argument to that in proof of Lemma 3.17 (ii) that when the function $f(\eta; a, \gamma)$ for $\eta \in (a, \eta_0]$ is Lipschitz continuous in γ and the Lipschitz constant is uniform in $\gamma \in [\gamma_0, \gamma_3]$, since $\eta = \eta_0$ is not special.

Now we prove $f(\eta_1; a, \gamma)$ is a continuous function when $\eta_1 > \eta_0$. We consider in two cases.

Case A. Consider $\gamma_0 > \gamma$. First, since f' is bounded for $\eta > a + \frac{\zeta}{2}$, we can choose a fixed η_1 such that $|f(\eta_0; a, \gamma_0) - f(\eta_1; a, \gamma_0)| < \frac{\delta}{2}$. We know there exists μ such that $|f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| < \frac{\delta}{2}$ for $|\gamma - \gamma_0| < \mu$. Then

$$\begin{aligned}&|f(\eta_0; a, \gamma) - f(\eta_1; a, \gamma_0)| \\ \leq& |f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| + |f(\eta_0; a, \gamma_0) - f(\eta_1; a, \gamma_0)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

Since $\gamma_0 > \gamma$, then by Lemma 3.24 (i) we have

$$f(\eta_1; a, \gamma_0) + \delta > f(\eta_0; a, \gamma) > f(\eta_1; a, \gamma) > f(\eta_1; a, \gamma_0),$$

so that $f(\eta_1; a, \gamma) - f(\eta_1; a, \gamma_0) < \delta$ if $\gamma_0 - \gamma < \mu$.

Case B. Now consider $\gamma_0 < \gamma$ and denote $f(\eta; a, \gamma) = f$ and $f(\eta; a, \gamma_0) = f_0$. We know that

$$-\frac{a}{2} f'(\eta) \leq \varepsilon[\phi'(-f(\eta))f'(\eta)]', \quad \eta > a,$$

letting $k > 1$ and $ka < \eta_0$, integrating (3.40) from ka to η yield

$$-\varepsilon\phi'(-f(\eta))f'(\eta) - \frac{a}{2} f(\eta) \leq -\varepsilon\phi'(-f(ka))f'(ka) - \frac{a}{2} f(ka).$$

Letting $\eta \to \eta_1$ gives

$$-f(\eta_1) < -\frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka) - f(ka). \tag{3.54}$$

Now consider the equation for f_0. Integrating from a to ka, we get

$$-\frac{ka}{2} f_0(ka) + \frac{1}{2}\int_a^{ka} f_0(s)\mathrm{d}s = \varepsilon\phi'(-f_0(ka))f_0(ka) + \gamma_0,$$

we know that $f_0(\eta_1) < f_0(\eta)$ for $\eta \in (\eta_0, \eta_1]$, then

$$f_0(\eta_1) < \frac{k}{k-1} f_0(ka) + \frac{2\varepsilon}{(k-1)a}\phi'(-f_0(ka))f_0'(ka) + \frac{2\gamma_0}{(k-1)a}. \tag{3.55}$$

Combining (3.54) and (3.55) we have

$$f_0(\eta_1) - f(\eta_1) < f_0(ka) - f(ka) + \frac{1}{k-1} f_0(ka) + \frac{2\gamma_0}{(k-1)a} - \frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka).$$

Choosing k such that $a < ka < \eta_1$ to satisfy $\dfrac{2\gamma_0}{(k-1)a} + \dfrac{f_0(ka)}{k-1} < \dfrac{\delta}{3}$. For $ka > a + \frac{\zeta}{2}$ we have $-f'(ka) < \dfrac{16\gamma}{4\zeta + \zeta^2}$ by (3.16). Now choose and fix k so that $-\dfrac{2\varepsilon}{a}\phi'(-f(ka))f'(ka) < \dfrac{\delta}{3}$. With this k, we know there exists $\mu > 0$ such that for $\gamma - \gamma_0 < \mu$

$$f_0(ka) - f(ka) < \frac{\delta}{3}.$$

Therefore $f(\eta_1; a, \gamma_0) - f(\eta_1; a, \gamma) < \delta$ if $\gamma - \gamma_0 \leq \mu$.

We can now conclude $f(\eta_1; a, \gamma)$ is a continuous function of γ for fixed a. It can be prove iteratively that $f(\eta; a, \gamma)$ is a continuous function of γ at fixed $\eta \in (a, \infty)$ with fixed a. Moreover, for fixed γ, $f(\eta; a, \gamma)$ is a continuous function of a can be proved by using the similar argument. $\square$

The following corollary is obtained directly from Lemma 3.26

Corollary 3.27. *If f satisfies (3.40) and (3.42), then*

(i) *$d(a, \gamma)$ is a continuous function of γ for fixed a;*

(ii) *$d(a, \gamma)$ is a continuous function of a for fixed γ.*

Now we prove that $d(a, \gamma)$ is a continuous function of both a and γ by using Lemma 3.24 (ii) and Lemma 3.25 (ii). The proof is different and longer than the proof of Lemma 3.19, since we consider $\eta \to \infty$ here.

Lemma 3.28. *$d(a, \gamma)$ is a continuous function of a and γ.*

Proof. If $d(a, \gamma)$ is continuous with a and γ, then for all $\delta > 0$ there exists $\mu > 0$ such that if $|(a, \gamma) - (a_0, \gamma_0)| < \mu$ then $|d(a, \gamma) - d(a_0, \gamma_0)| < \delta$.

Case 1. For $a > a_0$ and $\gamma < \gamma_0$, we choose a fixed η_0 such that $|f(\eta_0; a_0, \gamma_0) - d(a_0, \gamma_0)| < \frac{\delta}{2}$. We know from Lemma 3.26 that there exists μ such that $|f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| < \frac{\delta}{2}$ for $|(a, \gamma) - (a, \gamma_0)| < \frac{\mu}{2}$. Then

$$
\begin{aligned}
&|f(\eta_0; a, \gamma) - d(a_0, \gamma_0)| \\
\leq &|f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| + |f(\eta_0; a, \gamma_0) - d(a_0, \gamma_0)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}
$$

Since the sequence $a > a_0$ and $\gamma < \gamma_0$, then we have

$$d(a_0, \gamma_0) + \delta > f(\eta_0; a, \gamma) > d(a, \gamma) > d(a_0, \gamma_0),$$

then $d(a,\gamma) - d(a_0,\gamma_0) < \delta$ as $|(a,\gamma) - (a_0,\gamma_0)| < \mu$.

Case 2. For $a < a_0$, consider

$$|d(a,\gamma) - d(a_0,\gamma_0)| \leq |d(a,\gamma) - d(a_0,\gamma)| + |d(a_0,\gamma) - d(a_0,\gamma_0)|.$$

Given $\delta > 0$, there exist $\mu > 0$ such that $|d(a_0,\gamma) - d(a_0,\gamma_0)| < \frac{\delta}{2}$ if $|\gamma - \gamma_0| < \mu$, by Lemma 3.27 that $d(a,\gamma)$ is continuous in γ for fixed a.

Now considering $|d(a,\gamma) - d(a_0,\gamma)|$, we want to prove that $d(a,\gamma)$ is a continuous function of a uniformly in $\gamma \in [\gamma_0 - \mu, \gamma_0 + \mu]$ for $\mu > 0$. We denote $f(\eta; a,\gamma) = f$ and $f(\eta; a_0,\gamma) = f_0$. We know that

$$-\frac{a}{2}f'(\eta) \leq \varepsilon[\phi'(f(\eta))f'(\eta)]',$$

integrating (3.40) from ka to η where $k > 1$, we get

$$-\varepsilon\phi'(-f(\eta))f'(\eta) - \frac{a}{2}f(\eta) \leq -\varepsilon\phi'(-f(ka))f'(ka) - \frac{a}{2}f(ka).$$

Letting $\eta \to \infty$, by Lemma 3.7 we have

$$-d(a,\gamma) < -\frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka) - f(ka). \tag{3.56}$$

Now considering the equation of f_0 and integrating from a_0 to ka_0, we get

$$-\frac{ka_0}{2}f_0(ka_0) + \frac{1}{2}\int_{a_0}^{ka_0} f_0(\xi)\mathrm{d}\xi = \varepsilon\phi'(-f_0(ka_0))f_0'(ka_0) + \gamma,$$

and the lower bound $d(a_0,\gamma) < f_0$ and $\gamma \leq \gamma_0 + \mu$ then yields

$$d(a_0,\gamma) < \frac{k}{k-1}f_0(ka_0) + \frac{2\varepsilon}{(k-1)a_0}\phi'(-f_0(ka_0))f_0'(ka_0) + \frac{2(\gamma_0+\mu)}{(k-1)a_0}. \tag{3.57}$$

Combining (3.56) and (3.57), we have

$$\begin{aligned}&d(a_0,\gamma_0) - d(a,\gamma)\\ <&f_0(ka_0) - f(ka) + \frac{1}{k-1}f_0(ka) + \frac{2(\gamma_0+\mu)}{(k-1)a_0} - \frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka).\end{aligned}$$

Choose $k > 1$ such that $\dfrac{2(\gamma_0+\mu)}{(k-1)a_0} + \dfrac{1}{k-1} f_0(ka_0) < \dfrac{\hat{\delta}}{3}$ and for $ka > a+1$ we have $-f'(ka) < \dfrac{4(\gamma_0+\mu)}{2ka-1}$ by (3.16), so we can have $-\dfrac{2\varepsilon}{a}\phi'(-(ka))f'(ka) < \dfrac{\hat{\delta}}{3}$. With the choosing k we know that for $a < a_0$

$$f_0(ka_0) - f(ka) < f_0(ka_0) - f_0(ka) < \frac{\hat{\delta}}{3} \quad \text{as } |a_0 - a| < \mu.$$

Therefore, we have $d(a_0,\gamma) - d(a,\gamma) < \hat{\delta}$ with $\gamma \in [\gamma_0 - \mu, \gamma_0 + \mu]$. Then for given $\delta = 2\hat{\delta}$, there exists $\mu > 0$ such that $|d(a_0,\gamma) - d(a,\gamma)| < \frac{\delta}{2}$ if $|a - a_0| < \mu$ with $\gamma \in [\gamma_0 - \mu, \gamma_0 + \mu]$.

Case 3. For $\gamma_0 < \gamma$, we consider

$$|d(a,\gamma) - d(a_0,\gamma_0)| \le |d(a,\gamma) - d(a,\gamma_0)| + |d(a,\gamma_0) - d(a_0,\gamma_0)|.$$

Given $\delta > 0$, there exist $\mu > 0$ such that $|d(a,\gamma_0) - d(a_0,\gamma_0)| < \frac{\delta}{2}$ if $|a - a_0| < \mu$ by Lemma 3.27 that $d(a,\gamma)$ is a continuous function of a for fixed γ.

Now considering $|d(a,\gamma) - d(a,\gamma_0)|$, we want to prove that $d(a,\gamma)$ is a continuous function of γ uniformly with $a \in [a_0 - \mu, a_0 + \mu]$ for $\mu > 0$. We denote $f(\eta; a, \gamma) = f$ and $f(\eta; a, \gamma_0) = f_0$. We know that

$$-\frac{a}{2} f'(\eta) \le \varepsilon[\phi'(f(\eta))f'(\eta)]',$$

integrating (3.40) from ka to η where $k > 1$, we get

$$-\varepsilon\phi'(-f(\eta))f'(\eta) - \frac{a}{2} f(\eta) \le -\varepsilon\phi'(-f(ka))f'(ka) - \frac{a}{2} f(ka).$$

Letting $\eta \to \infty$, we have

$$-d(a,\gamma) < -\frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka) - f(ka). \tag{3.58}$$

Now considering the equation of f_0 and integrating from a to ka, we get

$$-\frac{ka}{2} f_0(ka) + \frac{1}{2}\int_a^{ka} f_0(\xi)\mathrm{d}\xi = \varepsilon\phi'(-f_0(ka))f_0(ka) + \gamma_0,$$

the lower bound $d(a,\gamma_0) < f_0$ yields

$$d(a,\gamma_0) < \frac{k}{k-1} f_0(ka) + \frac{2\varepsilon}{(k-1)a}\phi'(-f_0(ka))f_0'(ka) + \frac{2\gamma_0}{(k-1)a}. \qquad (3.59)$$

Combining (3.58) and (3.59) we have

$$d(a,\gamma_0) - d(a,\gamma) < f_0(ka) - f(ka) + \frac{1}{k-1} f_0(ka) + \frac{2\gamma_0}{(k-1)a} - \frac{2\varepsilon}{a}\phi'(-f(ka))f'(ka).$$

Since $a_0 - \mu < a < a_0 + \mu$, we can choose $k > 1$ such that $\frac{2(\gamma_0)}{(k-1)a_0} + \frac{1}{k-1} f_0(ka) < \frac{2(\gamma_0)}{(k-1)(a_0-\mu)} + \frac{1}{k-1} f_0(ka_0 - k\mu) < \frac{\hat{\delta}}{3}$ and for $ka > a+1$ we have $-f'(ka) < \frac{4\gamma}{2k(a_0-\mu)-1}$ by (3.16), so $-\frac{2\varepsilon}{a}\phi'(-(ka))f'(ka) < \frac{\hat{\delta}}{3}$ independently of a. With the choosing k we have

$$f(ka) - f_0(ka) < \frac{\hat{\delta}}{3} \quad \text{if } |\gamma_0 - \gamma| < \mu,$$

with $a \in [a_0 - \mu, a_0 + \mu]$, since f is a continuous function of γ for every η_0. Then with given $\delta = 2\hat{\delta}$, we have $|d(a,\gamma_0) - d(a,\gamma)| < \frac{\delta}{2}$. □

3.3 Two-parameter shooting method

In this section we will use two-parameter shooting to show that for each $U_0, V_0 > 0$, there exist $a, \gamma > 0$ such that the solution $f(\eta; a, \gamma)$ of (3.13) satisfies $b(a,\gamma) = U_0$ and $d(a,\gamma) = -V_0$.

We will use the following lemma which can be found in [15, Lemma 2.8] and [23, p.112].

Lemma 3.29. *Suppose that Λ_1 and Λ_2 are two connected open sets of $\mathbb{R}^2$, with components (maximal connected subset) $\tilde{\Lambda}_1 \subset \Lambda_1$ and $\tilde{\Lambda}_2 \subset \Lambda_2$ such that $\tilde{\Lambda}_1 \cap \tilde{\Lambda}_2$ is disconnected. Then $\Lambda_1 \cup \Lambda_2 \neq \mathbb{R}^2$.*

This result also applies to every subset of $\mathbb{R}^2$ which is homeomorphic to the entire plane [15, p.31]. We can apply it to the set $(0, \infty) \times (0, \infty)$, for example, if we define a homomorphism $g : (0, \infty) \times (0, \infty) \to \mathbb{R}^2$ such that $g(x, y) = (\log x, \log y)$.

Theorem 3.30. *Suppose $\varepsilon > 0$, then there exists a unique solution f of problem (3.13).*

Proof. First we identify four "bad" sets

$$\Gamma_1 = \left\{(a, \gamma) \,\middle|\, b(a, \gamma) > U_0\right\},$$

$$\Gamma_2 = \left\{(a, \gamma) \,\middle|\, b(a, \gamma) < U_0\right\},$$

$$\Gamma_3 = \left\{(a, \gamma) \,\middle|\, d(a, \gamma) > -V_0\right\},$$

$$\Gamma_4 = \left\{(a, \gamma) \,\middle|\, d(a, \gamma) < -V_0\right\}.$$

However, we do not have an appropriate topological lemma involving four sets. Therefore we combine the Γ_i to form two new sets, as follows:

$$\Lambda_1 = \Gamma_1 \cup \Gamma_4,$$

$$\Lambda_2 = \Gamma_2 \cup \Gamma_3.$$

It is easy to see that if (a, γ) is in $(0, \infty) \times (0, \infty)$ but not in $\Lambda_1 \cup \Lambda_2$, then we have a solution $f(\eta; a, \gamma)$ such that $b(a, \gamma) = U_0$ and $d(a, \gamma) = -V_0$. Now we want to show that Λ_1 and Λ_2 satisfy the hypothesis of Lemma 3.29.

These sets are clearly open in $(0, \infty) \times (0, \infty)$ since $b(a, \gamma)$ and $d(a, \gamma)$ are continuous functions of a and γ by Lemma 3.19 and 3.28. Γ_1 and Γ_2 are non-empty since $\lim_{a \to 0} b(a, \gamma) = 0$ and $\lim_{a \to \infty} b(a, \gamma) = \infty$ by Lemma 3.18. Moreover, $\lim_{a \to \infty} d(a, \gamma) = 0$ and $\lim_{\gamma \to \infty} d(a, \gamma) = -\infty$ by Lemma 3.24, yielding Γ_3 and Γ_4 are non-empty. Therefore, Λ_1 and Λ_2 are open and non-empty.

Lemma 3.31. *The sets Λ_1 and Λ_2 are connected.*

Proof. In the following, we will exploit the monotonicity of $b(a,\gamma)$ and $d(a,\gamma)$ in a and γ. First we prove that Γ_1, Γ_2, Γ_3 and Γ_4 are each connected. As an example, we prove that Γ_1 is connected. Given two points $(\tilde{a},\tilde{\gamma}),(\hat{a},\hat{\gamma}) \in \Lambda_1$, there are two cases:

(i). $\tilde{a} > \hat{a}$ and $\tilde{\gamma} \geq \hat{\gamma}$;

(ii). $\tilde{a} \geq \hat{a}$ and $\tilde{\gamma} < \hat{\gamma}$.

The following figures describe an admissible step path, contained in Γ_1, that connects $(\tilde{a},\tilde{\gamma})$ and $(\hat{a},\hat{\gamma})$ in each of two cases

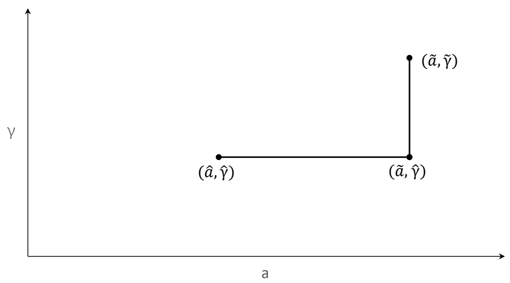

Figure 3.1: step-path of Γ_1(i)

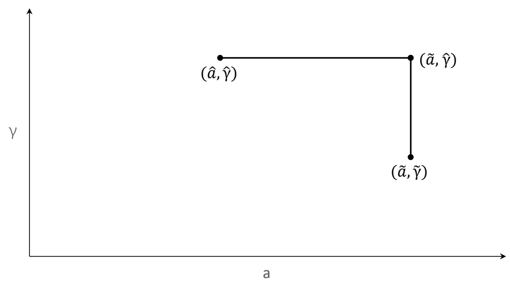

Figure 3.2: step-path of Γ_1(ii)

In Figure 3.1, if $\tilde{a} > \hat{a}$, then we have $(\tilde{a},\hat{\gamma}) \in \Gamma_1$, since $b(\tilde{a},\hat{\gamma}) > b(\hat{a},\hat{\gamma}) > U_0$ by Lemma 3.18 (i). It follows that the path connecting $(\tilde{a},\hat{\gamma})$ and $(\tilde{a},\tilde{\gamma})$ belongs to Γ_1, since $(\tilde{a},\hat{\gamma}),(\tilde{a},\tilde{\gamma}) \in \Gamma_1$ and $b(a,\gamma)$ is monotonically increasing in γ by Lemma 3.17 (i). Similarly, in Figure 3.2, $(\tilde{a},\hat{\gamma}) \in \Gamma_1$ as $\tilde{a} \geq \tilde{a}$, since $b(a,\gamma)$ is increasing in a by Lemma 3.18 (i), then the path connecting $(\tilde{a},\hat{\gamma})$ and $(\tilde{a},\tilde{\gamma})$ belongs to Γ_1 by Lemma 3.18 (i). We can prove by using a similar argument that Γ_2, Γ_3 and Γ_4 are each connected.

We now prove $\Gamma_1 \cap \Gamma_4$ and $\Gamma_2 \cap \Gamma_3$ are non-empty.

For fixed $a > 0$, since $\lim_{\gamma\to\infty} b(a,\gamma) = \infty$ by Lemma 3.17 (iii) and $\lim_{\gamma\to\infty} d(a,\gamma) = -\infty$ by Lemma 3.24 (ii) , we can find $\check{\gamma}$ large enough such that $b(a,\check{\gamma}) > U_0$ and $d(a,\check{\gamma}) < -V_0$. It follows that for $\check{\gamma}$ sufficiently large, $(a,\check{\gamma}) \in \Gamma_1 \cap \Gamma_4$, so $\Gamma_1 \cap \Gamma_4 \neq \emptyset$. Similarly, given $\widetilde{\gamma} > 0$, there exists $\widehat{a}$ small enough that $b(\widehat{a},\widetilde{\gamma}) < U_0$ since $\lim_{a\to 0} b(a,\gamma) = 0$ by Lemma 3.18 (ii). Then choose $\widehat{\gamma}$ smaller than $\widetilde{\gamma}$ if necessary to ensure that $d(\widehat{a},\widehat{\gamma}) > -V_0$ and $b(\widehat{a},\widehat{\gamma}) < U_0$ since $\lim_{\gamma\to 0} d(a,\gamma) = 0$ by Lemma 3.24 (iii) and $b(a,\gamma)$ is monotonically increasing in γ by Lemma 3.17 (i). It then follows that $(\widehat{a},\widehat{\gamma}) \in \Gamma_2 \cap \Gamma_3$, so $\Gamma_2 \cap \Gamma_3 \neq \emptyset$.

Then, since $\Gamma_1 \cap \Gamma_4 \neq \emptyset$, we can always find a point belonging to $\Gamma_1 \cap \Gamma_4$ that is path connected to both $(\hat{a},\hat{\gamma}) \in \Gamma_1$ and $(a^*,\gamma^*) \in \Gamma_4$, since Γ_1 and Γ_4 are each connected.

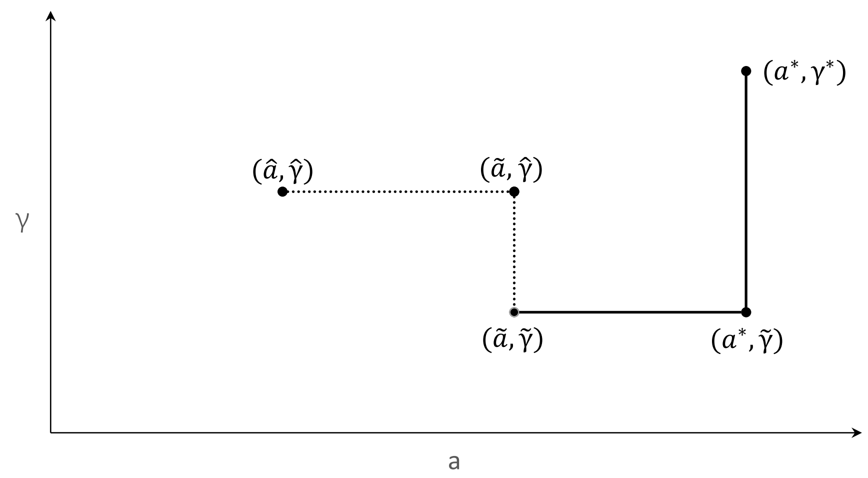

Figure 3.3: path-connectedness of Λ_1

For example, in Figure 3.3, the solid lines indicate that the path belongs to

Γ_1 and the dashed lines indicate that the path belongs to Γ_4. We can find $(\tilde{a},\tilde{\gamma}) \in \Gamma_1 \cap \Gamma_4$ since $\Gamma_1 \cap \Gamma_4 \neq \emptyset$. If $(\hat{a},\hat{\gamma}) \in \Gamma_4$ and $(a^*,\gamma^*) \in \Gamma_1$, then there are step paths each connecting $(\hat{a},\hat{\gamma})$ and $(\tilde{a},\tilde{\gamma})$, $(\tilde{a},\tilde{\gamma})$ and (a^*,γ^*), since Γ_1 and Γ_4 are each connected.

Therefore, Λ_1 is connected, and similarly, Λ_2 is connected since $\Gamma_2 \cap \Gamma_3 \neq \emptyset$ and Γ_2, Γ_3 are each connected. □

Now we take $\tilde{\Lambda}_1 = \Lambda_1$, $\tilde{\Lambda}_2 = \Lambda_2$.

Next we will show that $\Lambda_1 \cap \Lambda_2$ is disconnected. We have

$$\Lambda_1 \cap \Lambda_2 = (\Gamma_1 \cap \Gamma_2) \cup (\Gamma_1 \cap \Gamma_3) \cup (\Gamma_2 \cap \Gamma_4) \cup (\Gamma_3 \cap \Gamma_4).$$

Clearly $\Gamma_1 \cap \Gamma_2$, $\Gamma_3 \cap \Gamma_4$ are empty.

Lemma 3.32. *$\Lambda_1 \cap \Lambda_2$ is disconnected.*

Proof. For fixed γ, we can find $\tilde{a}$ large enough such that $b(\tilde{a},\gamma) > U_0$ and $d(\tilde{a},\gamma) > -V_0$, since $\lim\limits_{a\to\infty} b(a,\gamma) = \infty$ by Lemma 3.18 (iv) and $\lim\limits_{a\to\infty} d(a,\gamma) = 0$ by Lemma 3.25 (iii). It follows that for $\tilde{a}$ sufficient large, $(\tilde{a},\gamma) \in \Gamma_1 \cap \Gamma_3$, so $\Gamma_1 \cap \Gamma_3 \neq \emptyset$. Similarly, given $\hat{a} > 0$, there exits γ^* large enough such that $d(\hat{a},\gamma^*) < -V_0$, since $\lim\limits_{\gamma\to\infty} d(a,\gamma) = -\infty$ by Lemma 3.24 (ii). Then choose a^* smaller than $\hat{a}$ if necessary to ensure that $d(a^*,\gamma^*) < -V_0$ and $b(a^*,\gamma^*) < U_0$, since $\lim\limits_{a\to 0} b(a,\gamma) = 0$ by Lemma 3.18 (ii) and $d(a,\gamma)$ is monotonically decreasing in a by Lemma 3.25 (i). It then follows that $(a^*,\gamma^*) \in \Gamma_2 \cap \Gamma_4$, so $\Gamma_2 \cap \Gamma_4 \neq \emptyset$.

Therefore $\Gamma_1 \cap \Gamma_3$ and $\Gamma_2 \cap \Gamma_4$ are non-empty and disjoint. Therefore, $\Lambda_1 \cap \Lambda_2$ is disconnected since it is the union of non-empty, disjoint and open sets. □

Now Lemma 3.29 yields that there is a point $(\bar{a}, \bar{\gamma}) \in (0, \infty) \times (0, \infty)$ which is not in $\Lambda_1 \cup \Lambda_2$. Hence $b(\bar{a}, \bar{\gamma}) = U_0$ since $(\bar{a}, \bar{\gamma}) \notin \Gamma_1 \cup \Gamma_2$ and $d(\bar{a}, \bar{\gamma}) = -V_0$ since $(\bar{a}, \bar{\gamma}) \notin \Gamma_3 \cup \Gamma_4$. The result then follows from Theorem 2.34 and Theorem 3.3. $\square$

3.4 Self-similar solutions with $\varepsilon = 0$

Now we consider

$$-\frac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', \quad \eta < a, \tag{3.60}$$

with boundary conditions

$$f(0) = U_0, \tag{3.61}$$

$$\lim_{\eta \nearrow a} f(\eta) = 0, \quad \lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta) = -\frac{aV_0}{2}. \tag{3.62}$$

In Lemma 3.14, we showed that for each $a > 0, \gamma > 0$, there exists solution f for $\eta \in (a - \delta, a)$ for some $\delta > 0$. For $\varepsilon = 0$, we know that $f(\eta) = -V_0$ for $\eta > a$. Then with the special choice $\gamma = \frac{aV_0}{2}$, we obtain directly from Lemma 3.14 and Lemma 3.16 the following proposition.

Proposition 3.1. *For given a and γ, there exists $\delta > 0$ such that for $\eta \in (a - \delta, a)$, equation (3.60) has a unique solution which is positive and satisfies the boundary condition (3.62). This solution can be continuous back to $\eta = 0$.*

The following discussion on the behaviour of $f(\eta)$ as $\eta \to 0$ is analogous to that in Lemma 3.18. Note that $\gamma = \frac{aV_0}{2}$ when $\varepsilon = 0$.

Lemma 3.33. $b(a) := \lim_{\eta \to 0} f\left(\eta; a, \dfrac{aV_0}{2}\right)$ *has the following properties:*

(i) $b(a)$ *is strictly monotonically increasing in* a;

(ii) $\lim_{a\to 0} b(a) = 0$;

(iii) $b(a)$ *is a continuous function of* a;

(iv) $\lim_{a\to\infty} b(a) = \infty$;

Proof. (i) Denote $f_{a_i} = f(\eta; a_i, \gamma)$. Let f_{a_1} and f_{a_2} be positive solutions satisfying (3.26), (3.28) and corresponding to $a = a_1, a = a_2$. Suppose $b(a)$ is not strictly monotonically increasing in a. Then it is possible to find $0 < a_1 < a_2$ such that $b(a_2) \le b(a_1)$ and $\eta_0 \in (0, a_1)$ such that $f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$ and $f_{a_1} < f_{a_2}$ on (η_0, a_1), we denote $\bar{f} := f_{a_1}(\eta_0) = f_{a_2}(\eta_0)$.

Integrating the equation for f_{a_1} from η_0 to a_1 and the equation for f_{a_2} from η_0 to a_2, then subtracting these two equations, we obtain

$$\begin{aligned}\frac{(a_1 - a_2)V_0}{2} =& \frac{1}{2}\int_{\eta_0}^{a_1} (f_{a_2}(s) - f_{a_1}(s))\mathrm{d}s + \frac{1}{2}\int_{a_1}^{a_2} f_{a_2}(s)\mathrm{d}s \\ &+ \phi'(\bar{f})[f'_{a_2}(\eta_0) - f'_{a_1}(\eta_0)].\end{aligned}$$

Since $f_{a_1} < f_{a_2}$ on (η_0, a_1) and $f_{a_2} > 0$ on (a_1, a_2), the first and second term on the right-hand side are positive. The third term is non-negative because $f'_{a_2}(\eta_0) \ge f'_{a_1}(\eta_0)$ at η_0. The left-hand side is negative since $a_1 < a_2$. We therefore have a contradiction. The function $b(a, \gamma)$ must therefore be monotonically increasing in a.

(ii) Let $a < 1$, denote $N = \phi'(b(1))$, we have $\phi'(f) \le N$ by (i). Then we get from (3.60) that

$$\left\{ e^{\frac{\eta^2}{4N}} [\phi(f(\eta))]' \right\}' \ge 0,$$

and integrating from 0 to η then yields

$$[\phi(f(\eta))]' \ge Ae^{\frac{-\eta^2}{4N}},$$

where $A = \phi'(b(a))f'(0) < 0$. Integrating from η to a we get

$$\phi(f(\eta)) \leq -A \int_{\eta}^{a} e^{\frac{-s^2}{4N}} \mathrm{d}s.$$

Now we integrate the equation (3.26) from 0 to a and obtain

$$\frac{1}{2}\int_0^a f(s)\mathrm{d}s = \frac{aV_0}{2} - \phi'(b(a))f'(0).$$

Then $-A = \frac{aV_0}{2} + \frac{1}{2}\int_0^a f(s)\mathrm{d}s$ and the results follows from $-A \to 0$ as $a \to 0$.

The proof of (iii) is similar to the proof of Lemma 3.18 (iii), note that if $a \in (a_0, a_3)$, then $\gamma \in (\frac{a_0V_0}{2}, \frac{a_3V_0}{2})$.

We can obtain (iv) directly from Lemma 3.18 (iv) since $\gamma = \frac{aV_0}{2} > 0$ by Lemma 3.12. $\square$

Now, since γ can be express as a function of a, we will use one-parameter shooting to show the existence of self-similar solution.

Theorem 3.34. *Suppose $\varepsilon = 0$. Then there exists a unique solution f of problem (3.24).*

Proof. Identify two"bad" sets

$$S^- = \left\{a \,\middle|\, b(a) < U_0\right\},$$
$$S^+ = \left\{a \,\middle|\, b(a) > U_0\right\}.$$

We use a shooting method. Clearly S^- and S^+ are disjoint. By Lemma 3.33 we know that $b(a)$ is monotonically increasing, then we can find a large enough such that $b(a) > U_0$ since $\lim_{a\to\infty} b(a) = \infty$ and a small enough such that $b(a) < U_0$ since $\lim_{a\to 0} b(a) = 0$, yielding that S^- and S^+ are non-empty.

Moreover, S^- and S^+ are open since f is a continuous function of a. Indeed, let $a_0 \in S^-$ and let $\beta := U_0 - b(a_0)$, then there exists μ such that $|b(a) - b(a_0)| < \beta$ for $|a - a_0| < \mu$ since $b(a)$ is continuous by Lemma 3.33, which implies $b(a) < b(a_0) + \beta < U_0$. A similar proof shows that S^+ is open. Since S^- and S^+ are non-empty disjoint open sets, $S^- \cup S^+ \neq (0, \infty)$. Then we can conclude that there exists $a \notin S^- \cup S^+$, such that $b(a) = U_0$. The result follows from Theorem 2.34 and Theorem 3.11. $\square$

Chapter 4

The whole-line case: Problem (1.6) and self-similar solutions for the limit problems

In this chapter, we consider the problem (1.6) on the whole real-line. By using similar arguments to those used in Chapter 2 in the half-line case, we first prove the existence and uniqueness of the weak solution of (1.6) when $\varepsilon > 0$. We then prove some *a priori* bounds that will be used to study the $\varepsilon \to 0$ and $k \to \infty$ limits. Since the problems are now considered on $\mathbb{R}$, some alternative arguments and cut-off functions are used.

We will then prove the existence and uniqueness of the weak solution of the $k \to \infty$ limit problem and show that there exists a unique self-similar solution that is this weak solution of the limit problem. The existence of the self-similar solution when $\varepsilon > 0$ is proved by using a two-parameter shooting methods, similar to that used in Chapter 3 in the half-line case. Note that on the whole-line case, a, the position of the free boundary is not necessary positive. Therefore, in the study of the self-similar solutions, we consider three cases: $a < 0$, $a = 0$ and $a > 0$. We will also prove the existence of the self-similar solution when $\varepsilon = 0$ by a one-parameter shooting method. In this case, γ can be expressed as a function of a and V_0, therefore $a > 0$ since γ is positive. Our strategy takes advantage of ideas from Crooks and Hilhorst [10].

4.1 Existence and uniqueness of weak solutions for $\varepsilon > 0$

Let $\varepsilon > 0$. Similarly to the half-line case, we use an approximate problem to establish existence of solutions of (1.6). For each $R > 1$, let (4.1) denote the problem

$$\begin{cases} u_t = \phi(u)_{xx} - kuv, & (x,t) \in (-R,R) \times (0,T), \\ v_t = \varepsilon\phi(v)_{xx} - kuv, & (x,t) \in (-R,R) \times (0,T), \\ \phi(u)_x(-R,t) = \phi(u)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ \phi(v)_x(-R,t) = \phi(v)_x(R,t) = 0, & \text{for} \quad t \in (0,T), \\ u(x,0) = u^k_{0,R}(x), \quad v(x,0) = v^k_{0,R}(x), & \text{for} \quad x \in (-R,R), \end{cases} \tag{4.1}$$

where $u^k_{0,R}, v^k_{0,R} \in C^2(\mathbb{R}^+)$ are such that $0 \leq u^k_{0,R} \leq U_0$, $0 \leq v^k_{0,R} \leq V_0$ and

$$u^k_{0,R} = \begin{cases} U_0 - (U_0 - u^k_0)\hat{\psi}^R & x < 0, \\ u^k_0\hat{\psi}^R & x \geq 0, \end{cases} \qquad v^k_{0,R} = \begin{cases} v^k_0\hat{\psi}^R & x < 0, \\ V_0 - (V_0 - u^k_0)\hat{\psi}^R & x \geq 0, \end{cases} \tag{4.2}$$

which define the functions $u^k_{0,R}, v^k_{0,R}$ on the whole real line, where the family of cut-off functions $\hat{\psi}^R \in C^\infty(\mathbb{R}^+)$ with $R > 1$ are defined as

$$\hat{\psi}^R = \begin{cases} 1 & |x| \leq R, \\ \hat{\psi}^1(x + 1 - R) & |x| \geq R, \end{cases}$$

and $\hat{\psi}^1 \in C^\infty(\mathbb{R})$ is a even, non-negative cut-off function such that $0 \leq \hat{\psi}^1(x) \leq 1$ for all $x \in \mathbb{R}$, $\hat{\psi}^1(x) = 1$ when $|x| \leq 1$ and $\hat{\psi}^1(x) = 0$ when $|x| \geq 2$.

Now we introduce a notion of weak solution.

Definition 4.1. *A pair $(u_R^k, v_R^k) \in L^\infty((-R,R) \times (0,T)) \times L^\infty((-R,R) \times (0,T))$ is called a weak solution of (4.1) if*

(i) $\phi(u_R^k), \phi(v_R^k) \in L^2(0,T; W^{1,2}(-R,R))$;

(ii) (u_R^k, v_R^k) *satisfies*

$$\begin{aligned}
&\int_{-R}^{R} u_{0,R}^k \Psi(x,0)\mathrm{d}x + \int_0^T \int_{-R}^{R} u_R^k \Psi_t \mathrm{d}x\mathrm{d}t \\
=& \int_0^T \int_{-R}^{R} \phi(u_R^k)_x \Psi_x \mathrm{d}x\mathrm{d}t + k\int_0^T \int_{-R}^{R} \Psi u_R^k v_R^k \mathrm{d}x\mathrm{d}t, \\
&\int_{-R}^{R} v_{0,R}^k \Psi(x,0)\mathrm{d}x + \int_0^T \int_{-R}^{R} v_R^k \Psi_t \mathrm{d}x\mathrm{d}t \\
=& \int_0^T \int_{-R}^{R} \varepsilon\phi(v_R^k)_x \Psi_x \mathrm{d}x\mathrm{d}t + k\int_0^T \int_{-R}^{R} \Psi u_R^k v_R^k \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\Psi \in C^1([-R,R] \times [0,T])$ with $\Psi(\cdot, T) = 0$.

To prove the uniqueness of the weak solution of (4.1), we use the following comparison theorem. The proof is shown in a way similar to the proof of Lemma 2.2, replacing the spatial domain $(0,R)$ by $(-R,R)$.

Lemma 4.2. *Suppose $\varepsilon > 0$ and $(\overline{u}_R^k, \overline{v}_R^k)$, $(\underline{u}_R^k, \underline{v}_R^k)$ be such that*

(a) $\overline{u}_R^k, \underline{u}_R^k \in C((-R,R) \times (0,T))$;

(b) $\phi(\overline{u}_R^k), \phi(\underline{u}_R^k) \in L^2(0,T; W^{1,2}(-R,R))$,
$\overline{u}_{Rt}^k, \underline{u}_{Rt}^k, \phi(\overline{u}_R^k)_{xx}, \phi(\underline{u}_R^k)_{xx} \in L^1((-R,R) \times (0,T))$;

(c) $\overline{v}_R^k, \underline{v}_R^k \in C((-R,R) \times (0,T))$;

(d) $\phi(\underline{v}_R^k), \phi(\overline{v}_R^k) \in L^2(0,T; W^{1,2}(-R,R))$,
$\overline{v}_{Rt}^k, \underline{v}_{Rt}^k, \phi(\overline{v}_R^k)_{xx}, \phi(\underline{v}_R^k)_{xx} \in L^1((-R,R)\times(0,T))$;

$(\overline{u}_R^k, \overline{v}_R^k)$, $(\underline{u}_R^k, \underline{v}_R^k)$ satisfy

$$\begin{aligned}
&\overline{u}_{Rt}^k \geq \phi(\overline{u}_R^k)_{xx} - k\overline{u}_R^k\overline{v}_R^k, \quad \underline{u}_{Rt}^k \leq \phi(\underline{u}_R^k)_{xx} - k\underline{u}_R^k\underline{v}_R^k, && \text{in } (-R,R)\times(0,T),\\
&\overline{v}_{Rt}^k \leq \varepsilon\phi(\overline{v}_R^k)_{xx} - k\overline{u}_R^k\overline{v}_R^k, \quad \underline{v}_{Rt}^k \geq \varepsilon\phi(\underline{v}_R^k)_{xx} - k\underline{u}_R^k\underline{v}_R^k, && \text{in } (-R,R)\times(0,T),\\
&\overline{u}_R^k(-R,\cdot) \geq \underline{u}_R^k(-R,\cdot), \quad \phi(\overline{v}_R^k)_x(-R,\cdot) \leq \phi(\underline{v}_R^k)_x(-R,\cdot), && \text{on } (0,T),\\
&\phi(\overline{u}_R^k)_x(R,\cdot) \geq \phi(\underline{u}_R^k)_x(R,\cdot), \quad \phi(\overline{v}_R^k)_x(R,\cdot) \leq \phi(\underline{v}_R^k)_x(R,\cdot), && \text{on } (0,T),\\
&\overline{u}_R^k(\cdot,0) \geq \underline{u}_R^k(\cdot,0), \quad \overline{v}_R^k(\cdot,0) \leq \underline{v}_R^k(\cdot,0), && \text{on } (-R,R).
\end{aligned}$$

Then

$$\overline{u}_R^k \geq \underline{u}_R^k, \quad \overline{v}_R^k \leq \underline{v}_R^k \quad \text{in } (-R,R)\times(0,T).$$

We obtain the following immediately from Lemma 4.2.

Corollary 4.3. *Suppose $\varepsilon > 0$. For given initial data $u_{0,R}^k, v_{0,R}^k$, there is at most one solution (u_R^k, v_R^k) of (4.1).*

Corollary 4.4. *Let (u_R^k, v_R^k) be a weak solution of (4.1). Then we have*

$$0 \leq u_R^k(x,t) \leq U_0 \quad \text{and} \quad 0 \leq v_R^k(x,t) \leq V_0 \quad \text{for } (x,t) \in (-R,R)\times(0,T). \tag{4.3}$$

The existence of a weak solution of (4.1) is proved by an iterative method which is similar to that in the proof in half-line case, with boundary conditions $\phi(u_R^k)_x(R,t) = \phi(u_R^k)_x(-R,t) = 0$.

Theorem 4.5. *There exists a unique weak solution (u_R^k, v_R^k) of Problem (4.1) such that $0 \leq u_R^k \leq U_0$ and $0 \leq v_R^k \leq V_0$.*

The following results, analogous to those in Section 2.1 will yield existence of solution of (1.6) by passing to the limit $R \to \infty$ in (4.1).

Lemma 4.6. *Suppose $\varepsilon > 0$. Then for each $L > 0$ there exists a constant $C(L)$ such that if $R > L + 1$, then*

$$k \int_0^T \int_{-L-1}^{L+1} u_R^k v_R^k \mathrm{d}x\mathrm{d}t \leq C(L). \tag{4.4}$$

Proof. We recall a class of cut-off functions. First recall an even, non-negative cut-off function $\hat{\psi}^1 \in C^\infty(\mathbb{R})$ such that $0 \leq \hat{\psi}^1(x) \leq 1$ for all $x \in \mathbb{R}$

$$\hat{\psi}^1(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| \geq 2. \end{cases}$$

Then given $L \geq 1$, the family of cut-off function $\hat{\psi}^L \in C^\infty(\mathbb{R})$ is

$$\hat{\psi}^L(x) = \begin{cases} 1 & |x| \leq L, \\ \hat{\psi}^1(|x| + 1 - L) & |x| \geq L. \end{cases}$$

Clearly $\hat{\psi}^L, \hat{\psi}^L_x, \hat{\psi}^L_{xx}$ are bounded in $L^\infty(\mathbb{R})$ independently of L.

Multiplying the equation for u_R^k by cut-off function $\hat{\psi}^L$ and integrating over $(-R, R) \times (0, T)$ gives that

$$k \int_0^T \int_{-L-1}^{L+1} u_R^k v_R^k \hat{\psi}^L \mathrm{d}x\mathrm{d}t = \int_0^T \int_{-L-1}^{L+1} \phi(u_R^k) \hat{\psi}^L_{xx} \mathrm{d}x\mathrm{d}t - \int_{-L-1}^{L+1} \hat{\psi}^L u_R^k(x, T)\mathrm{d}x.$$

The fact that $0 \leq u_R^k \leq U_0$, together with definition of $\hat{\psi}^L$ and Lebesgue's Monotone Convergence Theorem yields (4.4). □

Lemma 4.7. *Suppose $\varepsilon > 0$. Then for each $L > 1$, $\phi(u_R^k), \phi(v_R^k)$ are bounded in $L^2\left(0, T; W^{1,2}_{loc}(\mathbb{R})\right)$ independently of k and R.*

Proof. This follows from the same form of argument used to show Lemma 2.11, multiplying the equation for u_R^k by $\phi(u_R^k)\hat{\psi}^L$ and integrating over $(-R,R)\times(0,T)$. Denoting $F=\int_0^{u_R^k}\phi(s)\mathrm{d}s$ which is bounded, then we have

$$\begin{aligned}\int_0^T\int_{-L-1}^{L+1}|\phi(u_R^k)_x|^2\hat{\psi}^L\mathrm{d}x\mathrm{d}t=&-\int_{-L-1}^{L+1}\big[F(x,T)-F(x,0)\big]\hat{\psi}^L\mathrm{d}x\\&+\frac{1}{2}\int_0^T\int_{-L-1}^{L+1}\big[\phi(u_R^k)\big]^2\hat{\psi}^L_{xx}\mathrm{d}x\mathrm{d}t\\&-k\int_0^T\int_{-L-1}^{L+1}\phi(u_R^k)\hat{\psi}^Lu_R^kv_R^k\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The results are yielded by Lemma 4.6 and (4.3). $\square$

Recall the notion for space and time translates introduced in (2.23). The following is the result of the gradient estimates in Lemma 4.6 and the proof is similar to the proof of Lemma 2.12, replacing integrals over $(r,L+1)$ with $(-L-1,L+1)$.

Lemma 4.8. *For each $L>0$, there exists a constant $C(L)$ such that*

$$\begin{aligned}\int_0^T\int_{-L-1}^{L+1}|\phi(S_\delta u_R^k)-\phi(u_R^k)|^2\mathrm{d}x\mathrm{d}t\le C(L)|\delta|^2,\\\int_0^T\int_{-L-1}^{L+1}|\phi(S_\delta v_R^k)-\phi(v_R^k)|^2\mathrm{d}x\mathrm{d}t\le C(L)|\delta|^2.\end{aligned}$$

The next result follows from arguments analogous to those used in the proof of Lemma 2.13, replacing ψ^L by $\hat{\psi}^L$ and integrals over $(0,R)$ with $(-R,R)$.

Lemma 4.9. *For each $L>0$, there exists a constant $C(L)$ such that*

$$\int_0^{T-\tau}\int_{-L-1}^{L+1}|\phi(T_\tau u_R^k)-\phi(u_R)|^2\mathrm{d}x\mathrm{d}t\le\tau C(L),$$

$$\int_0^{T-\tau}\int_{-L-1}^{L+1}|\phi(T_\tau v_R^k)-\phi(v_R)|^2\mathrm{d}x\mathrm{d}t\le\tau C(L).$$

Define the family of test functions

$$\hat{\mathcal{F}}_T:=\left\{\xi\in C^1(Q_T):\ \xi(\cdot,T)=0 \text{ for } t\in(0,T) \text{ and } \operatorname{supp}\xi\subset[-J,J]\times[0,T]\right.$$
$$\left.\text{for some } J>0\right\}.$$

Now we can establish the existence of a weak solution of the Problem (1.6) of Q_T when $\varepsilon>0$.

Theorem 4.10. *Suppose $\varepsilon>0$. Then for given $k>0$, there exists a weak solution $(u^k,v^k)\in(L^\infty(Q_T))^2$ of (1.6) such that for each $J>0$*

(i) $\phi(u^k)\in L^2(0,T;W^{1,2}((-J,J)))$, $\quad\phi(v^k)\in L^2(0,T;W^{1,2}(-J,J))$*;*

(ii) (u^k,v^k) *satisfies*

$$\int_{\mathbb{R}}u_0^k\Psi(x,0)\mathrm{d}x+\iint_{Q_T}u^k\Psi_t\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\phi(u^k)_x\Psi_x\mathrm{d}x\mathrm{d}t+k\iint_{Q_T}\Psi u^kv^k\mathrm{d}x\mathrm{d}t,$$
$$\int_{\mathbb{R}}v_0^k\Psi(x,0)\mathrm{d}x+\iint_{Q_T}v^k\Psi_t\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\varepsilon\phi(v^k)_x\Psi_x\mathrm{d}x\mathrm{d}t+k\iint_{Q_T}\Psi u^kv^k\mathrm{d}x\mathrm{d}t,$$

where $\Psi\in\hat{\mathcal{F}}_T$.

Proof. Let $u_{0,R},v_{0,R}$ be as in the formulation of problem (4.2) and note such that as $R\to\infty$, $u_{0,R}\to u_0^k$, $v_{0,R}\to v_0^k$ in $C^1_{loc}(\mathbb{R})$. Then given $R_n\to\infty$, it follows from the Fréchet-Kolmogorov Theorem, (4.1), Lemma 4.8 and Lemma 4.9, that there exist a subsequence $\left\{R_{n_j}\right\}_{j=1}^\infty$ and functions $u^k\in L^\infty(Q_T)$, $v^k\in L^\infty(Q_T)$ such that

$$u_{R_{n_j}}\to u^k,\quad v_{R_{n_j}}\to v^k \text{ strongly in } L^2_{loc}(Q_T) \text{ and a.e. in } Q_T$$

as $j \to \infty$. We know that $\{\phi(u_{R_n})\}$ and $\{\phi(v_{R_n})\}$ is bounded in $L^2(0,T;W^{1,2}(-J,J))$ by Lemma 4.7, so taking a further subsequence of necessary, we have that as $j \to \infty$

$$\begin{aligned} \phi(u_{R_{n_j}}) &\rightharpoonup \phi(u^k) \quad \text{in } L^2(0,T;W^{1,2}(-J,J)), \\ \phi(v_{R_{n_j}}) &\rightharpoonup \phi(v^k) \quad \text{in } L^2(0,T;W^{1,2}(-J,J)). \end{aligned}$$

By the Dominated Convergence Theorem we can then easily pass to the limit in the weak form of (1.6). □

We will use the following comparison principle to prove the uniqueness of the weak solution of Problem (1.6), this result covers both $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 4.11. *Suppose $\varepsilon \geq 0$ and let $(\overline{u}^k, \overline{v}^k)$, $(\underline{u}^k, \underline{v}^k)$ be such that*

(a) $\overline{u}^k, \underline{u}^k \in L^\infty(Q_T)$;

(b) $\phi(\overline{u}^k), \phi(\underline{u}^k) \in L^2(0,T;W^{1,2}(\mathbb{R}))$, $\overline{u}^k_t, \underline{u}^k_t, \phi(\overline{u}^k)_{xx}, \phi(\underline{u}^k)_{xx} \in L^1(Q_T)$;

(c) $\overline{v}^k, \underline{v}^k \in L^\infty(Q_T)$, $\overline{v}^k_t, \underline{v}^k_t \in L^1(Q_T)$;

(d) *If* $\varepsilon > 0$, $\phi(\overline{v}^k), \phi(\underline{v}^k) \in L^2(0,T;W^{1,2}(\mathbb{R}))$, $\phi(\overline{v}^k)_{xx}, \phi(\underline{v}^k)_{xx} \in L^1(Q_T)$;

$(\overline{u}^k, \overline{v}^k)$, $(\underline{u}^k, \underline{v}^k)$ *satisfy*

$$\begin{aligned} &\overline{u}^k_t \geq \phi(\overline{u}^k)_{xx} - k\overline{u}^k\overline{v}^k, \quad \underline{u}^k_t \leq \phi(\underline{u}^k)_{xx} - k\underline{u}^k\underline{v}^k, && \text{in } Q_T, \\ &\overline{v}^k_t \leq \varepsilon\phi(\overline{v}^k)_{xx} - k\overline{u}^k\overline{v}^k, \quad \underline{v}^k_t \geq \varepsilon\phi(\underline{v}^k)_{xx} - k\underline{u}^k\underline{v}^k, && \text{in } Q_T, \\ &\overline{u}^k(\cdot,0) \geq \underline{u}^k(\cdot,0), \quad \overline{v}^k(\cdot,0) \leq \underline{v}^k(\cdot,0), && \textit{on } \mathbb{R}. \end{aligned}$$

Then

$$\overline{u}^k \geq \underline{u}^k, \quad \overline{v}^k \leq \underline{v}^k \quad \textit{in } Q_T.$$

Proof. This follows from arguments analogous to those used in the proof of Lemma 2.15, replacing ψ^L by $\hat{\psi}^L$ and integrals over S_T by integrals over Q_T. $\square$

We obtain the following two corollaries immediately from Lemma 4.11.

Corollary 4.12. *Suppose $\varepsilon \geq 0$ and $k > 0$. Then for given initial data u_0^k, v_0^k, there is at most one solution (u^k, v^k) of (1.6).*

Corollary 4.13. *Let (u^k, v^k) be a weak solution of (1.6). Then we have*

$$0 \leq u^k(x,t) \leq U_0 \quad \text{and} \quad 0 \leq v^k(x,t) \leq V_0 \quad \text{for } (x,t) \in Q_T. \tag{4.5}$$

4.2 *A priori* bounds and existence of weak solutions for $\varepsilon = 0$

Again we begin with some preliminary estimates that will be used to prove the existence of the weak solution of (1.6) and to study the limit as $k \to \infty$, which are counterparts of results in Section 2.2. Here, some different arguments are needed because there is no longer a Dirichlet boundary condition at $x = 0$.

We adapt ideas and cut-off functions from [10, Lemma 2.12] to prove the following bound of ku^kv^k. Note that here, ku^kv^k is controlled by the u^k equation on $\mathbb{R}^+$ and by the v^k equation on $\mathbb{R}^-$.

Lemma 4.14. *There exists a constant $C > 0$, independent of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.6), we have*

$$\iint_{Q_T} ku^kv^k \mathrm{d}x\mathrm{d}t \leq C.$$

Proof. Define $\beta^1 \in C^\infty(\mathbb{R})$ such that $0 \leq \beta^1(x) \leq 1$ for all $x \in \mathbb{R}$,

$$\beta^1(x) = \begin{cases} 1 & x \in [0,1], \\ 0 & x \in (-\infty,-1] \cup [2,\infty). \end{cases}$$

Then given $L \geq 1$, the family of cut-off functions $\beta^L \in C^\infty(\mathbb{R})$ are defined by

$$\beta^L(x) = \begin{cases} \beta^1 & x < 0, \\ 1 & x \in [0, L], \\ \beta^1(x+1-L) & x \geq L. \end{cases}$$

Define $\tilde{\beta}^L \in C^\infty(\mathbb{R})$ by $\tilde{\beta}^L(x) = \beta^L(-x)$ for all $x \in \mathbb{R}$. Note that $0 \leq \beta^L(x), \tilde{\beta}^L(x) \leq 1$ for all $x \in \mathbb{R}$ and $\beta^L_x, \beta^L_{xx}, \tilde{\beta}^L_x, \tilde{\beta}^L_{xx}$ are bounded in both $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ independently of L.

Multiplying the equation for u^k by β^L and integrating over $\mathbb{R} \times (0, t_0)$ where $t_0 \in (0, T]$, give

$$\int_{\mathbb{R}} \beta^L u^k(x,t_0)\mathrm{d}x + k\int_0^{t_0}\int_{\mathbb{R}} \beta^L u^k v^k \mathrm{d}x\mathrm{d}t = \int_0^{t_0}\int_{\mathbb{R}} \beta^L_{xx}\phi(u^k)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \beta^L u^k_0(x)\mathrm{d}x,$$

which, by the definition of β^L, (4.5) and u^k_0 is bounded independently of k in $L^1(\mathbb{R}^+)$, imply that the right-hand side is bounded independently of L and k, given the existence of $C > 0$ such that for all $k > 0$ and $t_0 \in (0, T]$

$$\int_{-1}^{L+1} \beta^L u^k(x,t_0)\mathrm{d}x + k\int_0^{t_0}\int_{-1}^{L+1} \beta^L u^k v^k \mathrm{d}x\mathrm{d}t \leq C, \tag{4.6}$$

and then, letting $L \to \infty$ and using Lebesgue's monotone convergence theorem give

$$k\int_0^T\int_0^\infty u^k v^k \mathrm{d}x\mathrm{d}t \leq C. \tag{4.7}$$

Similarly, since $\{v^k_0\}$ is bounded independently of k in $L^1(\mathbb{R}^-)$, multiplying the equation for v^k by $\tilde{\beta}^L$ and integrating over $\mathbb{R} \times (0, t_0)$ yields that C

can be chosen large enough that for all L and $k > 0$, we have

$$\int_{-\infty}^{1} \tilde{\beta}^L v^k(x, t_0)\mathrm{d}x + k\int_0^{t_0}\int_{-\infty}^{1} \tilde{\beta}^L u^k v^k \mathrm{d}x\mathrm{d}t \leq C, \tag{4.8}$$

and hence, letting $L \to \infty$ yields that

$$k\int_0^T \int_{-\infty}^0 u^k v^k \mathrm{d}x\mathrm{d}t \leq C. \tag{4.9}$$

The result then follows from (4.7) and (4.9). □

The following L^1-bounds of $\{u^k(\cdot, t) - u_0^\infty\}$ and $\{v^k(\cdot, t) - v_0^\infty\}$ are proved similar to those in the proof of Lemma 2.19.

Lemma 4.15. *There exists a constant $C > 0$ independently of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.6), we have*

$$\|u^k(\cdot, t) - u_0^\infty\|_{L^1(\mathbb{R})} \leq C \text{ and } \|v^k(\cdot, t) - v_0^\infty\|_{L^1(\mathbb{R})} \leq C. \tag{4.10}$$

Proof. It first follows immediately from (4.6), (4.8) and Lebesgue's Monotone Convergence Theorem that there exists $C > 0$ independently of $\varepsilon \geq 0$ and $k > 0$, such that

$$\int_0^\infty u^k(x, t_0)\mathrm{d}x \leq C \quad \text{and} \quad \int_{-\infty}^0 v^k(x, t_0)\mathrm{d}x \leq C \quad \text{for all } t_0 \in [0, T]. \tag{4.11}$$

Now choose a smooth convex function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \geq 0,\ m(0) = 0,\ m'(0) = 0,\ m(r) = |r| - \frac{1}{2} \text{ for } |r| > 1,$$

and for each $\alpha > 0$, define the functions

$$m_\alpha(r) := \alpha m\left(\frac{r}{\alpha}\right)$$

which approximate the modulus function as $\alpha \to 0$, and define $\hat{u} = u^k - U_0$, $\hat{w} = \phi(u^k) - \phi(U_0)$, then $\hat{u}(x, 0) = u_0^k - U_0$.

Now with $\tilde{\beta}^L$ as in Lemma 4.14, multiplying the equation of $\hat{u}$ by $m'_\alpha(\hat{w})\tilde{\beta}^L$ and integrating over $\mathbb{R} \times (0, t_0)$, then letting $\alpha \to 0$ yields

$$\int_{\mathbb{R}} |\hat{u}(x, t_0) - \hat{u}(x, 0)|\tilde{\beta}^L \mathrm{d}x \leq \int_0^{t_0} \int_{\mathbb{R}} |\hat{w}|\tilde{\beta}^L_{xx} \mathrm{d}x\mathrm{d}t - k \int_0^{t_0} \int_{\mathbb{R}} \mathrm{sgn}(\hat{w})\tilde{\beta}^L u^k v^k \mathrm{d}x\mathrm{d}t,$$

then we have

$$\begin{aligned}
\int_{\mathbb{R}} |u^k(x, t_0) - U_0|\tilde{\beta}^L \mathrm{d}x &\leq \int_{\mathbb{R}} |u^k(x, t_0) - u^k(x, 0)|\tilde{\beta}^L \mathrm{d}x + \int_{\mathbb{R}} |u^k(x, 0) - U_0|\tilde{\beta}^L \mathrm{d}x \\
&\leq \int_{\mathbb{R}} |u^k(x, 0) - U_0|\tilde{\beta}^L \mathrm{d}x + \int_0^{t_0} \int_{\mathbb{R}} |\hat{w}|\tilde{\beta}^L_{xx} \mathrm{d}x\mathrm{d}t \\
&\quad - k \int_0^{t_0} \int_{\mathbb{R}} \mathrm{sgn}(\hat{w})\tilde{\beta}^L u^k v^k \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now by Lemma 4.14, (4.5) and the fact that $\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R})}$ is bounded independently of k, the right-hand side is bounded independently of L and k. So it follows that there exists C independent of k, such that

$$\int_{-\infty}^0 |u^k(x, t_0) - U_0| \mathrm{d}x \leq C, \quad \text{for all } t_0 \in (0, T]. \tag{4.12}$$

Then define $\hat{v} = v^k - V_0$, $\hat{z} = \phi(v^k) - \phi(V_0)$, taking β^L in Lemma 4.14, multiplying the equation of $\hat{v}$ by $m'_\alpha(\hat{z})\beta^L$ and again integrating over $\mathbb{R} \times (0, t_0)$ gives, using a similar argument to above, that C can be chosen large enough that we also have that

$$\int_0^{\infty} |v^k(x, t_0) - V_0| \mathrm{d}x \leq C, \quad \text{for all } t_0 \in (0, T]. \tag{4.13}$$

The result follows from (4.12), (4.13) and (4.11). $\square$

By using the Mean Value Theorem and (4.5), we obtain the following corollary.

Corollary 4.16. *There exists a constant $C > 0$ independently of $\varepsilon \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.6), we have*

$$\|\phi(u^k)(\cdot, t_0) - \phi(u_0^\infty)\|_{L^1(\mathbb{R})} \leq C \quad \text{and} \quad \|\phi(v^k)(\cdot, t_0) - \phi(v_0^\infty)\|_{L^1(\mathbb{R})} \leq C, \tag{4.14}$$

for all $t_0 \in [0, T]$.

Next, we prove the whole-line analogue of Lemma 2.21.

Lemma 4.17. *Suppose that ε 0. Then there exists $C > 0$, independent of $\varepsilon > 0$ and $k > 0$, such that for any solution (u^k, v^k) of (1.6),*

$$\iint_{Q_T} |\phi(u^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C, \quad \text{and} \quad \varepsilon \iint_{Q_T} |\phi(v^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C. \tag{4.15}$$

Proof. Let $\hat{\psi}^L$ be as in the proof of Lemma 4.6. Then multiplication of the equation for u^k by $\phi(u^k)\hat{\psi}^L$ and integration over Q_T gives

$$\begin{aligned}\iint_{Q_T} \phi(u^k)\hat{\psi}^L u_t^k \mathrm{d}x\mathrm{d}t = &- \iint_{Q_T} |\phi(u^k)_x|^2 \hat{\psi}^L \mathrm{d}x\mathrm{d}t + \frac{1}{2}\iint_{Q_T} |\phi(u^k)|^2 \hat{\psi}_{xx}^L \mathrm{d}x\mathrm{d}t \\ &- k\iint_{Q_T} \phi(u^k)\hat{\psi}^L u^k v^k \mathrm{d}x\mathrm{d}t,\end{aligned}$$

Now let $F = \displaystyle\int_0^{u^k} \phi(s)\mathrm{d}s$, then we have

$$\begin{aligned}\iint_{Q_T} |\phi(u^k)_x|^2 \hat{\psi}^L \mathrm{d}x\mathrm{d}t \leq &\int_{\mathbb{R}} \big|F(x,T) - F(x,0)\big|\hat{\psi}^L \mathrm{d}x + \frac{1}{2}\iint_{Q_T} |\phi(u^k)|^2 \hat{\psi}_{xx}^L \mathrm{d}x\mathrm{d}t \\ &- k\iint_{Q_T} \phi(u^k)\hat{\psi}^L u^k v^k \mathrm{d}x\mathrm{d}t.\end{aligned}$$

By using (4.5), we know that

$$\int_{\mathbb{R}} |F(x,T) - F(x,0)|\hat{\psi}^L \mathrm{d}x \leq \phi(U_0)(\|u^k(\cdot, T) - u_0^\infty\|_{L^1(\mathbb{R})} + \|u_0^\infty - u_0^k\|_{L^1(\mathbb{R})}), \tag{4.16}$$

which is bounded by Lemma 4.15. Combining with Lemma 4.14, using Lebesgue's Monotone Convergence Theorem and letting $L \to \infty$ implies that there exists a constant $C > 0$ such that

$$\iint_{Q_T} |\phi(u^k)_x|^2 \mathrm{d}x\mathrm{d}t \leq C,$$

independently of k. If $\varepsilon > 0$, the estimate for v^k_x can be proved likewise, using the equation for v^k. $\square$

Recall the notion for space and time translates introduced in (2.23). We now prove the estimates for the differences of space and time translates of solutions which will yield sufficient compactness to obtain the existence of solutions of (1.6) when $\varepsilon = 0$ and to study the $k \to \infty$ limit.

Lemma 4.18. *Suppose $\varepsilon \geq 0$ and let (u^k, v^k) be a solution of (1.6) satisfying (4.5). Then there exists a function $K \geq 0$ independent of $\varepsilon \geq 0$ and $k > 0$ such that $K(\delta) \to 0$ as $|\delta| \to 0$, and for $t \in (0, T]$,*

$$\int_{\mathbb{R}} \left|\phi(u^k) - \phi(S_\delta u^k)\right| + \left|\phi(v^k) - \phi(S_\delta v^k)\right| \mathrm{d}x \leq K(\delta).$$

Proof. Let u, v, w, z be as defined in (2.36), $u_0 := u^k_0 - S_\delta u^k_0$ and $v_0 := v^k_0 - S_\delta v^k_0$. Let $L, \alpha > 0$, $\hat{\psi}^L$ be the cut-off functions defined before in Lemma 4.6, and let m_α be as defined in the proof of Lemma 4.15. Then multiplying the equation for u by $\hat{\psi}^L m'_\alpha(w)$ and integrating over $\mathbb{R} \times (0, t_0), t_0 \in (0, T]$, gives

$$\begin{aligned}\int_0^{t_0} \int_{\mathbb{R}} m'_\alpha(w) \hat{\psi}^L u_t \mathrm{d}x\mathrm{d}t \leq & \int_0^{t_0} \int_{\mathbb{R}} m_\alpha(w) \left(\hat{\psi}^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ & - k \int_0^{t_0} \int_{\mathbb{R}} m'_\alpha(w) \hat{\psi}^L \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t,\end{aligned}$$

letting $\alpha \to 0$ and by [14, Lemma 7.6] which says (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}} \hat{\psi}^L |u(x,t_0) - u_0| \mathrm{d}x \leq \int_0^{t_0} \int_{\mathbb{R}} |w| \left(\hat{\psi}^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ - k \int_0^{t_0} \int_{\mathbb{R}} \hat{\psi}^L \mathrm{sgn}(w) \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t,\end{aligned}$$

then we have

$$\begin{aligned}\int_{\mathbb{R}} \hat{\psi}^L |u(x,t_0)| \mathrm{d}x &\leq \int_{\mathbb{R}} \hat{\psi}^L |u(x,t_0) - u_0| \mathrm{d}x + \int_{\mathbb{R}} \hat{\psi}^L |u_0| \mathrm{d}x \\ &\leq \int_{\mathbb{R}} \hat{\psi}^L |u_0| \mathrm{d}x + \int_0^{t_0} \int_{\mathbb{R}} |w| \left(\hat{\psi}^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ &\quad - k \int_0^{t_0} \int_{\mathbb{R}} \hat{\psi}^L \mathrm{sgn}(w) \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.17}$$

and similarly

$$\begin{aligned}\int_{\mathbb{R}} \hat{\psi}^L |v(x,t_0)| \mathrm{d}x \leq \int_{\mathbb{R}} \hat{\psi}^L |v_0| \mathrm{d}x + \int_0^{t_0} \int_{\mathbb{R}} |z| \left(\hat{\psi}^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ - k \int_0^{t_0} \int_{\mathbb{R}} \hat{\psi}^L \mathrm{sgn}(z) \left(u^k v^k - S_\xi u^k S_\xi v^k\right) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.18}$$

Adding (4.17) and (4.18) then gives

$$\begin{aligned}&\int_{\mathbb{R}} \hat{\psi}^L \left(|u(x,t_0)| + |v(x,t_0)|\right) \mathrm{d}x \\ &\leq \int_{\mathbb{R}} \hat{\psi}^L \left(|u_0| + |v_0|\right) \mathrm{d}x + \int_0^{t_0} \int_{\mathbb{R}} (|w| + |z|) \left(\hat{\psi}^L\right)_{xx} \mathrm{d}x\mathrm{d}t \\ &\quad - k \int_0^{t_0} \int_{\mathbb{R}} \hat{\psi}^L [\mathrm{sgn}(w) + \mathrm{sgn}(z)] \left(u^k v^k - S_\delta u^k S_\delta v^k\right) \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Letting $L \to \infty$, the fact that

$$[\mathrm{sgn}(w) + \mathrm{sgn}(z)] \left(u^k v^k - S_\delta u^k S_\delta v^k\right) > 0$$

together with (4.5) give that for each $t_0 \in (0,T)$,

$$\int_{\mathbb{R}} |u^k(x,t_0) - u^k(x+\delta,t_0)| + |v^k(x,t_0) - v^k(x+\delta,t_0)|\mathrm{d}x$$
$$\leq \int_{\mathbb{R}} |u_0^k(x) - u_0^k(x+\delta)| + |v_0^k(x) - v_0^k(x+\delta)|\mathrm{d}x.$$

The existence of K is then immediate from the assumption that

$$\|u_0^k(\cdot+\delta) - u_0^k(\cdot)\|_{L^1(\mathbb{R})} + \|v_0^k(\cdot+\delta) - v_0^k(\cdot)\|_{L^1(\mathbb{R})} \leq \omega(\delta)$$

where $\omega(\delta) \to 0$ as $\delta \to 0$. $\square$

The follows from arguments analogous to those used in the proof of Lemma 2.23, replacing ψ^L by $\hat{\psi}^L$ and integrals over $\mathbb{R}^+$ by integrals over $\mathbb{R}$.

Lemma 4.19. *Suppose $\varepsilon \geq 0$ and let (u^k, v^k) be a solution of (1.6) satisfying (4.5). Then there exists $C > 0$, independent of ε and k, such that for any $\tau \in (0,T)$,*

$$\int_0^{T-\tau} \int_{\mathbb{R}} |\phi(T_\tau u^k) - \phi(u^k)|^2 \mathrm{d}x\mathrm{d}t \leq \tau C,$$
$$\int_0^{T-\tau} \int_{\mathbb{R}} |\phi(T_\tau v^k) - \phi(v^k)|^2 \mathrm{d}x\mathrm{d}t \leq \tau C.$$

We can now prove a convergence result for solution (u^k, v^k) of (1.6) as $\varepsilon \to 0$.

Lemma 4.20. *Let $k > 0$ be fixed and $(u_\varepsilon^k, v_\varepsilon^k)$ be solution of (1.6) satisfying (4.5) with $\varepsilon > 0$. Then there exist $(u_\star^k, v_\star^k) \in (L^\infty(Q_T))^2$ such that up to a subsequence, for each $J > 0$*

$$\phi(u_\varepsilon^k) \to \phi(u_\star^k) \qquad \text{in} \quad L^2((0,J)\times(0,T)),$$
$$u_\varepsilon^k \to u_\star^k \qquad \text{a.e. in} \quad (0,J)\times(0,T),$$

$$\begin{aligned}
\phi(v_\varepsilon^k) \to \phi(v_\star^k) \qquad & \text{in} \quad L^2((0,J)\times(0,T)),\\
v_\varepsilon^k \to v_\star^k \qquad & \text{a.e. in} \quad (0,J)\times(0,T),\\
\phi(u_\varepsilon^k)-\phi(\tilde u) \rightharpoonup \phi(u_\star^k)-\phi(\tilde u) \qquad & \text{in} \quad L^2\left(0,T;W^{1,2}(\mathbb{R})\right),
\end{aligned}$$

as $\varepsilon \to 0$, where $\tilde u \in C^\infty(\mathbb{R})$ is a smooth function such that $\tilde u = u_0^\infty$ for all $|x|>1$.

Proof. It follows from Corollary 4.13 and 4.16 that $\{\phi(u_\varepsilon^k)-\phi(u_0^\infty)\}$ and $\{\phi(v_\varepsilon^k)-\phi(v_0^\infty)\}$ are bounded independently of $\varepsilon \geq 0$ in $L^2(Q_T)$. By Lemma 4.18 and Lemma 4.19, using the Riesz-Fréchet-Kolmogorov Theorem [3, Theorem 4.26], yields that the sets $\left\{\phi(v_\varepsilon^k)-\phi(v_0^\infty)\right\}_{\varepsilon>0}$ and $\left\{\phi(u_\varepsilon^k)-\phi(u_0^\infty)\right\}_{\varepsilon>0}$ are each relatively compact in $L^2\left((-J,J)\times(0,T)\right)$ for each $J>0$. The weak convergence of $\phi(u_\varepsilon^k)-\phi(\hat u)$ in $L^2\left(0,T;W^{1,2}(\mathbb{R})\right)$ follows from the fact that $\phi(u_\varepsilon^k)-\phi(u_0^\infty)$ is bounded independently of $\varepsilon \geq 0$ in $L^2(Q_T)$ together with the proof of Lemma 4.17. Then we know that $\phi(u_\varepsilon^k)\to\phi(u_\star^k)$ and $\phi(v_\varepsilon^k)\to\phi(v_\star^k)$ almost everywhere in $(-J,J)\times(0,T)$, so since ϕ^{-1} is continuous, then we have $u_\varepsilon^k \to \phi^{-1}(\phi(u_\star^k))$ and $v_\varepsilon^k \to \phi^{-1}(\phi(v_\star^k))$ almost everywhere in $(-J,J)\times(0,T)$. $\square$

Lemma 4.20 and Corollary 4.3 enable the following result to be established using arguments similar to those that yield Theorem 2.26. We omit details of the proof. Recall

$$\hat{\mathcal{F}}_T := \left\{\xi \in C^1(Q_T):\ \xi(\cdot,T)=0 \text{ for } t\in(0,T) \text{ and } \operatorname{supp}\xi \subset [-J,J]\times[0,T] \text{ for some } J>0\right\}.$$

Theorem 4.21. *Let $\varepsilon = 0$ and $k>0$. Then Problem (1.6) has a unique weak solution $(u^k,v^k)\in(L^\infty(Q_T))^2$ such that*

(i) $\phi(u^k)\in L^2(0,T;W^{1,2}((-J,J)))$;

(ii) (u^k, v^k) *satisfies*

$$\int_{\mathbb{R}} u_0^k \Psi(x,0)\mathrm{d}x + \iint_{Q_T} u^k \Psi_t \mathrm{d}x\mathrm{d}t = \iint_{Q_T} \phi(u^k)_x \Psi_x \mathrm{d}x\mathrm{d}t + k \iint_{Q_T} \Psi u^k v^k \mathrm{d}x\mathrm{d}t,$$
$$\int_{\mathbb{R}} v_0^k \Psi(x,0)\mathrm{d}x + \iint_{Q_T} v^k \Psi_t \mathrm{d}x\mathrm{d}t = k \iint_{Q_T} \Psi u^k v^k \mathrm{d}x\mathrm{d}t,$$

for all $\Psi \in \hat{\mathcal{F}}_T$.

4.3 The limit problem for (1.6) as $k \to \infty$

The next result follows directly from arguments similar to those used in Section 2.3, exploiting the whole-line estimates established in Section 4.2.

Lemma 4.22. *Let* $\varepsilon \geq 0$ *be fixed and* (u^k, v^k) *be solutions of (1.4) satisfying (4.5) with* $k > 0$. *Then there exists* $(u, v) \in (L^\infty(Q_T))^2$ *such that up to a subsequence, for each* $J > 0$

$$\begin{aligned}
\phi(u^k) &\to \phi(u) && \text{in} \quad L^2((-J,J)\times(0,T)),\\
u^k &\to u && \text{a.e. in} \quad (-J,J)\times(0,T),\\
\phi(v^k) &\to \phi(v) && \text{in} \quad L^2((-J,J)\times(0,T)),\\
v^k &\to v && \text{a.e. in} \quad (-J,J)\times(0,T),\\
\phi(u^k) - \phi(\tilde{u}) &\rightharpoonup \phi(u) - \phi(\tilde{u}) && \text{in} \quad L^2\left(0,T; W^{1,2}(\mathbb{R})\right),
\end{aligned}$$

and for $\varepsilon > 0$

$$\phi(v^k) - \phi(\tilde{v}) \rightharpoonup \phi(v) - \phi(\tilde{v}) \qquad \text{in} \quad L^2\left(0,T; W^{1,2}(\mathbb{R})\right),$$

as $k \to \infty$, *where* $\tilde{u}, \tilde{v} \in C^\infty(\mathbb{R})$ *are smooth functions such that* $\tilde{u} = u_0^\infty, \tilde{v} = v_0^\infty$ *for all* $|x| > 1$. *Moreover*

$$uv = 0 \text{ a.e. } \textit{in } Q_T. \tag{4.19}$$

Taking w^k and w as

$$w^k := u^k - v^k, \quad w := u - v, \tag{4.20}$$

we clearly again have that as a sequence $k_n \to \infty$, $w^{k_n} \to w$ in $L^2(Q_T)$ and almost everywhere in Q_T, and that

$$u = w^+, \ v = w^-.$$

The following result focus on the function $u - v$, which is useful on the derivation of the limit problem.

Lemma 4.23. *Let $\varepsilon \geq 0$ and (u, v) be as in Lemma 4.22. Then*

$$\iint_{Q_T} (u - v)\Psi_t \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} (u_0^\infty - v_0^\infty)\Psi(x, 0)\mathrm{d}x = \iint_{Q_T} [\phi(u)_x - \varepsilon\phi(v)_x] \Psi_x \mathrm{d}x\mathrm{d}t,$$

for all $\Psi \in \hat{\mathcal{F}}_T$.

Proof. Multiplying the difference between the equations for u^k and v^k by $\Psi \in \hat{\mathcal{F}}_T$ and integrating over Q_T gives

$$-\iint_{Q_T} (u^k - v^k)\Psi_t \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}} (u_0^k - v_0^k)\Psi(x, 0)\mathrm{d}x = \iint_{Q_T} \left[\phi(u^k)_x - \varepsilon\phi(v^k)_x\right] \Psi_x \mathrm{d}x\mathrm{d}t,$$

the result follows using Lemma 4.22 and the fact that $u_0^k \to u_0^\infty$ and $v_0^k \to v_0^\infty$ as $k \to \infty$. $\square$

Now recall the definition of $\mathcal{D}$ from (2.51) and define the limit problem

$$\begin{cases} w_t = \mathcal{D}(w)_{xx}, & \text{in } Q_T, \\ w(x, 0) = w_0(x) := \begin{cases} U_0, & \text{if } x < 0, \\ -V_0, & \text{if } x > 0. \end{cases} \end{cases} \tag{4.21}$$

Definition 4.24. *A function w is a weak solution of problem (4.21) if*

(i) $w \in L^\infty(Q_T)$,

(ii) $\mathcal{D}(w) \in \mathcal{D}(\hat{w}) + L^2(0,T;W^{1,2}(\mathbb{R}))$, *where* $\hat{w} \in C^\infty(\mathbb{R})$ *is a smooth function with* $\hat{w} = U_0$ *when* $x < -1$ *and* $\hat{w} = -V_0$ *when* $x > 1$,

(iii) w *satisfies for all* $T > 0$

$$\int_{\mathbb{R}} w_0 \Psi(x,0)\mathrm{d}x + \iint_{Q_T} w\Psi_t \mathrm{d}x\mathrm{d}t = \iint_{Q_T} \mathcal{D}(w)_x \Psi_x \mathrm{d}x\mathrm{d}t, \tag{4.22}$$

for all $\Psi \in \hat{\mathcal{F}}_T$.

Theorem 4.25. *The function* w *defined in (4.20) is a weak solution of problem (4.21) and the whole sequence* (u^k, v^k) *in Lemma 4.22 converges to* $(w^+, -w^-)$.

Proof. The existence of a weak solution is a straight forward consequence of Definition 4.24 and Lemma 4.23. The fact that the whole sequence (u^k, v^k) converges to $(w^+, -w^-)$ follows from the uniqueness result proved in Theorem 4.26 below. $\square$

Next, we prove the uniqueness of the weak solution of problem (4.21) for both $\varepsilon > 0$ and $\varepsilon = 0$.

The following theorem can be proved by using the similar arguments to those in the proof of Theorem 2.32.

Theorem 4.26. *Let* $\varepsilon > 0$, *then there exist a unique weak solution* w *of the limit problem (4.21).*

Note that the proof of Theorem 4.26 does not apply in the case $\varepsilon = 0$, since $\mathcal{D}(w_1) - \mathcal{D}(w_2) = 0$ for $w_1, w_2 < 0$. We therefore need an alternative method to prove the uniqueness of weak solution. The next follows from arguments analogous to those used in Theorem 2.33, replacing spatial domain $\mathbb{R}^+$ by $\mathbb{R}$.

Theorem 4.27. *Let $\varepsilon \geq 0$ and consider two solutions $w, \tilde{w}$ of problem (4.21) with initial data $w_0, \tilde{w}_0$ respectively, then*

$$\iint_{Q_T} |w - \tilde{w}| \mathrm{d}x\mathrm{d}t \leq C(T) \int_{\mathbb{R}} |w_0 - \tilde{w}_0| \mathrm{d}x, \tag{4.23}$$

and there exists at most one solution of problem (4.21) for given initial function w_0.

By Theorem 4.25, Theorem 4.26 and Theorem 4.27, we obtain the following.

Theorem 4.28. *Let $\varepsilon \geq 0$, then there exists a unique solution w of the limit problem (4.21).*

Next, we will prove that (4.21) has a self-similar solution, which is the unique solution, since we proved the uniqueness of the weak solution of (4.21) in Theorem 4.28. As in the half-line case, we can identify the limit w as a certain self-similar solution both when $\varepsilon > 0$ and when $\varepsilon = 0$. We first state the analogue of Corollary 3.2.

Proposition 4.1. *Let w be the unique weak solution of problem (4.21). Suppose that there exists a function $\beta : [0, T] \to \mathbb{R}$ such that for each $t \in [0, T]$*

$$w(x,t) > 0 \text{ if } x < \beta(t) \quad \text{and} \quad w(x,t) < 0 \text{ if } x > \beta(t).$$

Then if $t \mapsto \beta(t)$ is sufficiently smooth and the functions $u := w^+$ and $v := w^-$ are smooth up to $\beta(t)$, the function u, v satisfy one of two limit *problems, depending on whether $\varepsilon > 0$ or $\varepsilon = 0$. If $\varepsilon > 0$, then*

$$\begin{cases} u_t = \phi(u)_{xx}, & \text{in } \{(x,t) \in Q_T : x < \beta(t)\}, \\ v = 0, & \text{in } \{(x,t) \in Q_T : x < \beta(t)\}, \\ v_t = \varepsilon\phi(v)_{xx}, & \text{in } \{(x,t) \in Q_T : x > \beta(t)\}, \\ u = 0, & \text{in } \{(x,t) \in Q_T : x > \beta(t)\}, \\ \lim\limits_{x \nearrow \beta(t)} u(x,t) = 0 = \lim\limits_{x \searrow \beta(t)} v(x,t) & \text{for each } t \in [0,T], \\ \lim\limits_{x \nearrow \beta(t)} \phi[u(x,t)]_x = -\varepsilon \lim\limits_{x \searrow \beta(t)} \phi[v(x,t)]_x & \text{for each } t \in [0,T], \\ u(\cdot,0) = u_0^\infty, & \text{in } \mathbb{R} \\ v(\cdot,0) = v_0^\infty, & \text{in } \mathbb{R}, \end{cases} \tag{4.24}$$

whereas if $\varepsilon = 0$ and we suppose additionally that $\beta(0) = 0$ and $t \mapsto \beta(t)$ is a non-decreasing function, then

$$\begin{cases} u_t = \phi(u)_{xx}, & \text{in } \{(x,t) \in Q_T : x < \beta(t)\}, \\ v = 0, & \text{in } \{(x,t) \in Q_T : x < \beta(t)\}, \\ v = V_0, & \text{in } \{(x,t) \in Q_T : x > \beta(t)\}, \\ u = 0, & \text{in } \{(x,t) \in Q_T : x > \beta(t)\}, \\ \lim\limits_{x \nearrow \beta(t)} u(x,t) = 0 & \text{for each } t \in [0,T], \\ V_0\beta'(t) = -\lim\limits_{x \nearrow \beta(t)} \phi[u(x,t)]_x & \text{for each } t \in [0,T], \\ u(\cdot,0) = u_0^\infty, & \text{in } \mathbb{R} \\ v(\cdot,0) = v_0^\infty, & \text{in } \mathbb{R}, \end{cases} \tag{4.25}$$

where $\beta'(t)$ denotes the speed of propagation of the free boundary $\beta(t)$ and we suppose that $\beta(0) = 0$ and $t \mapsto \beta(t)$ is a non-decreasing function.

4.4 Self-similar solutions for the limit problems

In this section, we will prove that if we have a self-similar solution of (4.29), then it is a weak solution of (4.21).

The following results will be used to prove that if there is a self-similar solution, then it is a weak solution of (4.21) and will be useful later in Chapter 5.

Lemma 4.29. *If f satisfies (4.32) and boundary condition (4.33), (4.34), then for $\eta < \min\{a, 0\}$ we have*

$$\phi(U_0) - \phi(f(\eta)) \leq G \int_{-\infty}^{\eta} e^{\frac{-s^2}{4\phi'(U_0)}} \mathrm{d}s, \tag{4.26}$$

where

$$G = \begin{cases} -\phi'(f(0))f'(0), & \text{if } a > 0, \\ \gamma, & \text{if } a \leq 0. \end{cases}$$

Proof. Denote $N = \phi'(U_0)$, we have $\phi'(f(\eta)) \leq N$. Then we get directly from the equation of η for $\eta < 0$ that

$$-\frac{\eta}{2N}[\phi(f(\eta))]' \geq [\phi(f(\eta))]'',$$

then multiplying by $e^{\frac{\eta^2}{4N}}$ we get

$$\left\{ e^{\frac{\eta^2}{4N}} [\phi(f(\eta))]' \right\}' \leq 0, \tag{4.27}$$

for $\eta < 0 < a$, integrating from η to 0 yields

$$[\phi(f(\eta))]' \geq C e^{\frac{-\eta^2}{4N}},$$

where $C = \phi'(f(0))f'(0) < 0$. Integrating again from $-\infty$ to η we get

$$\phi(U_0) - \phi(f(\eta)) \leq -C \int_{-\infty}^{\eta} e^{\frac{-s^2}{4N}} \mathrm{d}s.$$

Similarly, for $\eta < a \leq 0$, integrating (4.27) from η to a yields

$$[\phi(f(\eta))]' \geq \gamma e^{\frac{-\eta^2}{4N}}.$$

Integrating again from $-\infty$ to η we get

$$\phi(U_0) - \phi(f(\eta)) \leq \gamma \int_{-\infty}^{\eta} e^{\frac{-s^2}{4N}} \mathrm{d}s.$$

□

We can get similar estimates to those in Lemma 4.29 for comparison of $\phi(f)$ to $\phi(V_0)$ as $\eta \to \infty$ together with Lemma 3.8. Then we have the following corollary.

Corollary 4.30. *If f satisfies (5.2), then f converges to $U_0, -V_0$ exponentially as η tends to $-\infty, \infty$.*

Proof. We know from Lemma 4.29 that, for $\eta < 0$

$$\phi'(s)\big(U_0 - f(\eta)\big) \leq G \int_{-\infty}^{\eta} e^{\frac{-s^2}{4\phi'(U_0)}} \mathrm{d}s, \tag{4.28}$$

for some η such that $f(\eta) < s < U_0$. Since $f(\eta) \to U_0$ as $\eta \to -\infty$, there exists a $\eta_0 < -1$ such that $f(\eta) > \frac{U_0}{2}$ as $\eta < \eta_0$. Then we have if $\eta < \eta_0$, $\phi'(s) > \phi'(\frac{U_0}{2})$, because $\frac{U_0}{2} < \eta < U_0$. It follows from (4.28) that for $\eta < \eta_0$

$$U_0 - f(\eta) \leq \frac{G}{\phi'(\frac{U_0}{2})} \int_{-\infty}^{\eta} e^{\frac{-s^2}{4\phi'(U_0)}} \mathrm{d}s \leq K e^{\frac{\eta}{4\phi'(U_0)}},$$

where $K = \frac{4G\phi'(U_0)}{\phi'(\frac{U_0}{2})}$.

The proof for f converges to $-V_0$ exponentially as $\eta \to \infty$ can be proved similarly. □

Theorem 4.31. *The unique weak solution w of problem (4.21) with $\varepsilon > 0$ has a self-similar form. There exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that*

$$w(x,t) = f(\frac{x}{\sqrt{t}}),\ (x,t) \in Q_T \text{ and } \beta(t) = a\sqrt{t},\ t \in [0,T].$$

Denote $\eta = \dfrac{x}{\sqrt{t}}$, f satisfies the system

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', & \text{if } \eta < a, \\ -\dfrac{1}{2}\eta f'(\eta) = [\varepsilon\phi'(-f(\eta))f'(\eta)]', & \text{if } \eta > a, \\ \lim\limits_{\eta\to-\infty} f(\eta) = U_0, \quad \lim\limits_{\eta\to\infty} f(\eta) = -V_0, \\ \lim\limits_{\eta\nearrow a} f(\eta) = 0 = -\lim\limits_{\eta\searrow a} f(\eta), \\ \lim\limits_{\eta\nearrow a} \phi'(f(\eta))f'(\eta) = \varepsilon \lim\limits_{\eta\searrow a} \phi'(-f(\eta))f'(\eta). \end{cases} \tag{4.29}$$

where a prime denotes differentiation with respect to η.

Proof. The proof is similar to the half-line case, Theorem 3.3, together with Corollary 4.30. The existence of solution of Problem (4.29) is proved in Theorem 4.50. $\square$

Theorem 4.32. *The unique weak solution w of problem (4.21) with $\varepsilon = 0$ has a self-similar form. There exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ and a constant $a \in \mathbb{R}^+$ such that*

$$w(x,t) = f(\frac{x}{\sqrt{t}}),\ (x,t) \in Q_T \text{ and } \beta(t) = a\sqrt{t},\ t \in [0,T].$$

Denote $\eta = \dfrac{x}{\sqrt{t}}$*, f satisfies the system*

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', & \text{if } \eta < a, \\ f(\eta) = -V_0, & \text{if } \eta > a, \\ \lim\limits_{\eta\to-\infty} f(\eta) = U_0, \\ \lim\limits_{\eta\nearrow a} f(\eta) = 0, \\ \lim\limits_{\eta\nearrow a} \phi'(f(\eta))f'(\eta) = -\dfrac{aV_0}{2}, \end{cases} \tag{4.30}$$

where a prime denotes differentiation with respect to η*.*

We now study the existence of a solution f that satisfies (4.29). Similar to the half-line problem, we split the proof into two parts: $\eta < a$ where $f(\eta) > 0$, and $\eta > a$ where $f(\eta) < 0$. We will also discuss the existence and properties of $\lim\limits_{\eta\to-\infty} f(\eta)$ and $\lim\limits_{\eta\to\infty} f(\eta)$. The main difference with the half-line case is that now we need to consider $\eta \in \mathbb{R}$ and investigate the case when $a \le 0$ in addition to $a > 0$.

We start with some preliminary results that will be used later. The monotonicity of f follows from arguments analogous to those in the proof of Lemma 3.4.

Lemma 4.33. *Suppose $\varepsilon > 0$. If f satisfies (4.29), then $f'(\eta) < 0$ for all $\eta \neq a$.*

Now we prove γ is strictly positive when $\varepsilon > 0$. Recall that when $\varepsilon > 0$

$$\gamma = -\lim_{\eta\searrow a} \phi'(f(\eta))f'(\eta) = -\lim_{\eta\nearrow a} \phi'(-f(\eta))f'(\eta).$$

Lemma 4.34. *Suppose $\varepsilon > 0$. Let f be a solution of (4.29), then $\gamma > 0$.*

Proof. Suppose $\gamma \le 0$, we consider in two cases, $a \ge 0$ and $a \le 0$. When $a \ge 0$, the proof is the same to the proof of Lemma 3.5.

Now we let $a \leq 0$. Integrating the equation for $\eta < a$ in (4.29) from η to a yields

$$-\frac{1}{2}\int_{\eta}^{a} sf'(s)\mathrm{d}s = -\phi'(f(\eta))f'(\eta) - \gamma. \tag{4.31}$$

The left-hand side of (4.31) is negative since $\eta < 0$ and $f'(\eta) < 0$ by Lemma 4.33 whereas the right-hand side of (4.31) is positive if $\gamma \leq 0$ since $f'(\eta) < 0$. Therefore, it follows by contradiction that $\gamma > 0$. $\square$

The following lemma proves the analogous result for γ when $\varepsilon = 0$. Recall that in this case, $\gamma = \lim_{\eta \searrow a} \phi'(f(\eta))f'(\eta) = \dfrac{aV_0}{2}$.

Lemma 4.35. *Suppose $\varepsilon = 0$ and let f be a solution of (4.30). Then $a, \gamma > 0$.*

Proof. We know from the proof of Lemma 4.34 that $\gamma > 0$ when $a \leq 0$, since the proof when $a \leq 0$ only involved the equation (4.31) for $\eta < a$. However, when $\varepsilon = 0$, the fact that $\gamma = \frac{aV_0}{2} > 0$ contracts $a \leq 0$. In conclusion, if f satisfies (4.30) when $\varepsilon = 0$, both a and γ are positive. $\square$

4.5 Self-similar solutions with $\varepsilon > 0$

4.5.1 $f > 0$ case for $\eta < a$

First we consider f that satisfies the equation

$$-\frac{1}{2}\eta f'(\eta) = [\phi'(f(\eta))f'(\eta)]', \quad \eta < a. \tag{4.32}$$

At the boundaries we require

$$\lim_{\eta \to -\infty} f(\eta) = U_0, \tag{4.33}$$

$$\lim_{\eta \nearrow a} f(\eta) = 0, \quad \lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta) = -\gamma, \tag{4.34}$$

where a and $\gamma > 0$ are constant.

In fact, we can get some results when $a < 0$ from the half-line problem by a change of variables. We define

$$-g(-\eta) := f(\eta), \tag{4.35}$$

denote $\hat{a} = -a$ and $\hat{\eta} = -\eta$, we get

$$-\frac{1}{2}\hat{\eta}g'(\hat{\eta}) = [\phi'(-g(\hat{\eta}))g'(\hat{\eta})]'.$$

From the previous results in half-line case, we know immediately that: the solution f exists is unique locally in a left-neighbourhood $(a-\delta, a)$ of a when $a > 0$, and it is monotonically decreasing. By the change of variables (4.35), we know from Lemma 3.21 and 3.22 that solution f exists is unique locally in a left-neighbourhood $(a-\delta, a)$ of a when $a < 0$.

We therefore have the following lemma, for which it remains to prove the local existence and uniqueness of the solution f when $a = 0$.

Lemma 4.36. *For given $a \in \mathbb{R}$ and $\gamma > 0$, there exists $\delta > 0$ such that in $(a-\delta, a)$ equation (3.40) has a unique solution which is positive and satisfies the boundary condition (4.34).*

Proof. The proof for $a = 0$ is similar to the proof of Lemma 3.14 and 3.16. If $a = 0$, integrating (4.32) from η to a yields

$$\frac{1}{f'(\eta)} = \frac{2\phi'(f(\eta))}{\int_0^f sf'(s)\mathrm{d}s + 2\gamma}. \tag{4.36}$$

We now treat η as a function of f, writing $\eta = \sigma(f)$, then (4.36) takes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}f} = \frac{2\phi'(f(\eta))}{\int_0^f \sigma(s)\mathrm{d}s + 2\gamma}.$$

Integrating from 0 to $f$gives

$$\sigma(f) = -2\int_0^f \frac{\phi'(\theta)}{\int_0^\theta \sigma(s)\mathrm{d}s + 2\gamma}\mathrm{d}\theta, \tag{4.37}$$

and if we set

$$\tau(f) = -\sigma(f) = -\eta,$$

then (4.37) becomes

$$\tau(f) = 2\int_0^f \frac{\phi'(\theta)}{-\int_0^\theta \tau(s)\mathrm{d}s + 2\gamma}\mathrm{d}\theta. \tag{4.38}$$

Now we denote by X the set of continuous functions $\tau(f)$ on $[0,\mu]$, satisfying $0 \leq \tau(f) \leq \frac{1}{2}$, and $\|\cdot\|$ the supremum norm on X. Then X is a complete metric space. Choose μ small enough that $\mu < 2\gamma$, on X we introduce the map

$$M(\tau)(f) = 2\int_0^f \frac{\phi'(\theta)}{-\int_0^\theta \tau(s)\mathrm{d}s + 2\gamma}\mathrm{d}\theta \leq 2\int_0^\mu \frac{\phi'(\theta)}{\gamma}\mathrm{d}\theta.$$

It is clear that $M(\tau)(f)$ is well-defined, non-negative and continuous. Moreover, $M(\tau)(f) \leq \frac{1}{2}$ if

$$\int_0^\mu \frac{\phi'(\theta)}{\gamma}\mathrm{d}\theta \leq \frac{1}{4}. \tag{4.39}$$

Therefore, if μ is chosen small enough that (4.39) is satisfied, M maps X into itself.

We wish to ensure that M is a contraction map, so let $\tau_1, \tau_2 \in X$, we have for chosen $\mu < 2\gamma$

$$
\begin{aligned}
\|M(\tau_1) - M(\tau_2)\| \le & 2\int_0^f \frac{\phi'(\theta)\int_0^\theta |\tau_1(s) - \tau_2(s)|\mathrm{d}s}{[-\int_0^\theta \tau_1(s)\mathrm{d}s + 2\gamma][-\int_0^\theta \tau_2(s)\mathrm{d}s + 2\gamma]}\mathrm{d}\theta \\
\le & 2\int_0^\mu \frac{\phi'(\theta)\theta}{\gamma^2}\mathrm{d}\theta\|\tau_1 - \tau_2\| \\
\le & 4\int_0^\mu \frac{\phi'(\theta)}{\gamma}\mathrm{d}\theta\|\tau_1 - \tau_2\|,
\end{aligned}
$$

and it follows that M is a contraction map if

$$
4\int_0^\mu \frac{\phi'(\theta)}{\gamma}\mathrm{d}\theta < 1.
$$

This constitutes our third restriction on μ, which implies the first one (4.39). The result follows from a contraction mapping principle [12]. $\square$

We know that if $a > 0$, the local solution in Lemma 4.36 can be continued back to $\eta = 0$ from Lemma 3.16 in the half-line case. The following bound for $f'(0)$ ensures f can be continued back a little bit from 0 by Picard's theorem and the fact that $f(0) > 0$.

Lemma 4.37. *If $a > 0$, f satisfies (4.32) and the boundary conditions (4.33) (4.34), then*

$$
-f'(0) \le \frac{af(0) + 2\gamma}{2\phi'(f(0))}. \tag{4.40}
$$

Proof. Integrating (4.32) from η to a we get

$$
\frac{a}{2}f(\eta) \ge -\gamma - \phi'(f(\eta))f'(\eta), \tag{4.41}
$$

Now let $\eta = 0$ we have

$$
-f'(0) \le \frac{af(0) + 2\gamma}{2\phi'(f(0))}.
$$

$\square$

In order to show that the unique local solution in Lemma 4.36 can be continued back to $\eta = -\infty$ by the Global Picard Theorem, we will prove some estimates for $-f'$ and f.

The following lemma proves the boundedness for f in three cases: $a > 0$, $a = 0$ and $a < 0$.

Lemma 4.38. *If f satisfies (4.32) and the boundary conditions (4.33) and (4.34), then we have for fixed a, γ, there exists $K > 0$ such that $0 < f(\eta) < K$ for all $\eta < a$.*

Proof. **Case 1.** For $a > 0$, first we consider $\eta \in [0, a)$, from (4.41) we know that

$$\frac{-\phi'(f(\eta))f'(\eta)}{\frac{2\gamma}{a} + f(\eta)} \leq \frac{a}{2}, \tag{4.42}$$

integrating (4.42) from 0 to η gives

$$\int_0^{f(0)} \frac{\phi'(s)}{\frac{2\gamma}{a} + f(s)} \mathrm{d}s \leq \frac{a^2}{2},$$

then $f(\eta) \leq f(0)$ is bounded for $0 \leq \eta < a$, since f is monotonic decreasing in η.

Next we consider $\eta \in [-2\rho, 0)$ for some positive ρ. Integrating (4.32) from η to 0 we have

$$\phi'(f(0))f'(0) - \phi'(f(\eta))f'(\eta) \leq 0, \tag{4.43}$$

then integrating (4.43) yields

$$\phi(f(\eta)) \leq \phi(f(0)) + 2\rho\phi'(f(0))f'(0).$$

Then for $\eta < -\rho$, integrating (4.32) from η to $-\rho$ we get

$$\frac{\rho}{2}\int_{\eta}^{-\rho} f'(s)\mathrm{d}s \geq \phi'(f(-\rho))f'(-\rho) - \phi'(f(\eta))f'(\eta),$$

so since $f' < 0$, we have

$$\frac{\rho}{2}f(\eta) - \phi'(f(\eta))f'(\eta) \leq \frac{\rho}{2}f(-\rho) - \phi'(f(-\rho))f'(-\rho),$$

therefore $f(\eta) \leq K$ for fixed a, γ and all $\eta < a$.

Case 2. For $a = 0$, the same proof can be used as in $a > 0$ case, with $\phi(f(\eta)) \leq 2\rho\gamma$ for $\eta \in [-2\rho, a)$.

Case 3. For $a < 0$, if $\eta \in [-2\rho, a)$, integrating (4.32) from η to a we have

$$-(\phi(f(\eta)))' < \gamma, \tag{4.44}$$

then integrating (4.44) from η to a gives

$$\phi(f(\eta)) \leq (a + 2\rho)\gamma.$$

For $\eta < -\rho$ the proof is the same as $a > 0$. $\square$

Next, we prove the boundedness for $-f'$ in three cases: $a > 0$, $a = 0$ and $a < 0$.

Lemma 4.39. *If f satisfies (4.32) and the boundary conditions (4.33) (4.34), then for fixed a, γ, there exists $\hat{K}$ such that $0 < -\phi'(f(\eta))f'(\eta) < \hat{K}$ for all $\eta < a$.*

Proof. For all $a \in \mathbb{R}$, first consider $\eta \in [-2\rho, a)$ for some $\rho > 0$. Integrating (4.32) from η to a we have that $-\phi'(f(\eta))f'(\eta) \leq \gamma + K(a + \rho)$, where K is positive constant that $f(\eta) \leq K$, by Lemma 4.38. Then for $\eta \leq -2\rho$, we know that $-\phi'(f(\eta))f'(\eta) \leq -\phi'(f(-2\rho))f'(-2\rho) \leq \hat{K}$ for fixed a, γ. $\square$

The following lemma follows from by Lemma 4.38, Lemma 4.39 together with [6, Theorem 1.186].

Lemma 4.40. *For given a, γ, the unique local solution in Lemma 4.36 can be continued back to $\eta = -\infty$.*

Now define

$$b(a, \gamma) := \lim_{\eta \to -\infty} f(\eta; a, \gamma),$$

where $\gamma := -\lim_{\eta \nearrow a} \phi'(f(\eta))f'(\eta)$ with $\gamma > 0$. Note that we use the same notation $b(a, \gamma)$ as in the half-line case, but here $b(a, \gamma)$ define as the function of $f(\eta; a, \gamma)$ as $\eta \to -\infty$ rather than $\eta \to 0$.

We can obtain from Corollary 3.20, Lemma 3.26 and the change of variables (4.35) that f is a continuous function of a and γ.

Lemma 4.41. *For each fixed $\eta^* < a$, if f satisfies (4.32) and (4.34), then*

(i) *$f(\eta^*; a, \gamma)$ is a continuous function of γ for fixed a;*

(ii) *$f(\eta^*; a, \gamma)$ is a continuous function of a for fixed γ.*

The following corollary follows from directly from Lemma 4.41 as $\eta \to -\infty$.

Corollary 4.42. *If f satisfies (4.32) and (4.34), then*

(i) *$b(a, \gamma)$ is a continuous function of γ for fixed a;*

(ii) *$b(a, \gamma)$ is a continuous function of a for fixed γ.*

Next we discuss the properties of $b(a, \gamma)$. When we study the properties of $b(a, \gamma) = \lim_{\eta \to -\infty} f(\eta; a, \gamma)$, we can see the properties of $d_{\mathbb{R}^+}(a, \gamma) = \lim_{\eta \to \infty} f_{\mathbb{R}^+}(\eta; a, \gamma)$ in half-line case.

We obtain the following from Lemma 3.7 by the change of variable (4.35).

Lemma 4.43. *If f satisfies (4.32) and the boundary conditions (4.33) and (4.34), then we have the derivative of f vanishes as $\eta \to -\infty$*

$$\lim_{\eta\to-\infty} f'(\eta) = 0.$$

Lemma 4.44. *$b(a,\gamma)$ has the following properties with fixed γ:*

(i) $b(a,\gamma)$ *is strictly monotonically increasing of* a;

(ii) $\lim_{a\to-\infty} b(a,\gamma) = 0$;

(iii) $\lim_{a\to\infty} b(a,\gamma) = \infty$.

Proof. The proof for (i),(iv) is the same with Lemma 3.18.

(ii) Consider $a < 0$, integrating (4.32) from η to a we get

$$-\phi'(f(\eta))f'(\eta) \leq \gamma - \frac{a}{2}\int_\eta^a f'(s)\mathrm{d}s = \frac{a}{2}\left(\frac{2\gamma}{a} + f(\eta)\right),$$

then the result follows from the fact that $f' < 0$, gives $\frac{2\gamma}{a} + f < 0$. □

The following lemma follows from the same argument as in the proof of Lemma 3.17 (ii).

Lemma 4.45. *$b(a,\gamma)$ has the following properties with fixed a:*

(i) $b(a,\gamma)$ *is strictly monotonically increasing in* γ;

(ii) $\lim_{\gamma\to\infty} b(a,\gamma) = \infty$;

(iii) $\lim_{\gamma\to 0} b(a,\gamma) = 0$.

Lemma 4.46. *$b(a, \gamma)$ is a continuous function of γ and a.*

Proof. The continuity result for $a < 0$ follows from Lemma 3.28 by the change of variable (4.35). Now consider $a \geq 0$, if $b(a, \gamma)$ is continuous with a and γ, then for all $\delta > 0$ there exists $\mu > 0$ such that if $|(a, \gamma) - (a_0, \gamma_0)| < \mu$ then $|b(a, \gamma) - b(a_0, \gamma_0)| < \delta$.

Case 1. For $a < a_0$ and $\gamma < \gamma_0$, we choose a fixed η_0 such that $|f(\eta_0; a, \gamma_0) - b(a, \gamma_0)| < \frac{\delta}{2}$. We know there exists μ such that $|f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| < \frac{\delta}{2}$ for $|(a, \gamma) - (a, \gamma_0)| < \frac{\mu}{2}$ by Lemma 4.41. Then

$$\begin{aligned}&|f(\eta_0; a, \gamma) - b(a_0, \gamma_0)| \\ \leq &|f(\eta_0; a, \gamma) - f(\eta_0; a, \gamma_0)| + |f(\eta_0; a, \gamma_0) - b(a_0, \gamma_0)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

Since the sequence $a < a_0$ and $\gamma < \gamma_0$, then we have

$$b(a_0, \gamma_0) > b(a, \gamma) > f(\eta_0; a, \gamma) > b(a_0, \gamma_0) - \delta,$$

then $b(a_0, \gamma_0) - b(a, \gamma) < \delta$ as $|(a_0, \gamma_0) - (a, \gamma)| < \mu$.

Case 2. For $a > a_0$, consider

$$|b(a, \gamma) - b(a_0, \gamma_0)| \leq |b(a, \gamma) - b(a_0, \gamma)| + |b(a_0, \gamma) - b(a_0, \gamma_0)|.$$

Given $\delta > 0$, there exist $\mu > 0$ such that $|b(a_0, \gamma) - b(a_0, \gamma_0)| < \frac{\delta}{2}$ if $|\gamma - \gamma_0| < \mu$ by Lemma 4.45 (ii) $b(a, \gamma)$ is continuous of γ for fixed a.

Now consider $|b(a, \gamma) - b(a_0, \gamma)|$, we want to prove that $b(a, \gamma)$ is a continuous function of a uniformly with $\gamma \in [\gamma_0 - \mu, \gamma_0 + \mu]$ for $\mu > 0$. We denote $f(\eta; a, \gamma) = f$ and $f(\eta; a_0, \gamma) = f_0$. We know from above that

$$b(a, \gamma) - b(a_0, \gamma) < f(-\rho) - f_0(-\rho) - \frac{2}{\rho}\phi'(f(-\rho))f'(-\rho) - \frac{2}{a_0}\phi'(f_0(-\rho))f_0'(-\rho).$$

We know that $f'(-\rho)$ is bounded, choosing ρ such that $-\frac{2}{\rho}\phi'(f(-\rho))f'(-\rho)-\frac{2}{a_0}\phi'(f_0(-\rho))f_0'(-\rho) < \frac{\hat{\delta}}{2}$. With this ρ there exists $\mu > 0$ that $|a - a_0| < \mu$

$$f(-\rho) - f_0(-\rho) < \frac{\hat{\delta}}{2},$$

then $|b(a,\gamma) - b(a_0,\gamma)| < \hat{\delta}$ with $\gamma \in [\gamma_0 - \mu, \gamma_0 + \mu]$. For given $\delta = 2\hat{\delta}$, we get the result.

Case 3. For $\gamma_0 < \gamma$, consider

$$|b(a,\gamma) - b(a_0,\gamma_0)| \leq |b(a,\gamma) - b(a,\gamma_0)| + |b(a,\gamma_0) - b(a_0,\gamma_0)|.$$

Given $\delta > 0$, there exist $\mu > 0$ such that $|b(a,\gamma_0)-b(a_0,\gamma_0)| < \frac{\delta}{2}$ if $|a-a_0| < \mu$ by Lemma 4.44 (iii) $b(a,\gamma)$ is continuous of a for fixed γ.

Now consider $|b(a,\gamma)-b(a,\gamma_0)|$, we want to prove that $b(a,\gamma)$ is a continuous function of γ uniformly with $a \in [a_0 - \mu, a_0 + \mu]$ for $\mu > 0$. We denote $f(\eta; a,\gamma) = f$ and $f(\eta; a, \gamma_0) = f_0$. We know that

$$\begin{aligned} b(a,\gamma) - b(a_0,\gamma) <& f(-\rho) - f_0(-\rho) - \frac{2}{\rho}\phi'(f(-\rho))f'(-\rho) \\ & - \frac{2}{a_0}\phi'(f_0(-\rho))f_0'(-\rho). \end{aligned}$$

We know that $f'(-\rho)$ is bounded, choosing ρ such that $-\frac{2}{\rho}\phi'(f(-\rho))f'(-\rho)-\frac{2}{a_0}\phi'(f_0(-\rho))f_0'(-\rho) < \frac{\hat{\delta}}{2}$. With this ρ there exists $\mu > 0$ that $|\gamma - \gamma_0| < \mu$

$$f(-\rho) - f_0(-\rho) < \frac{\hat{\delta}}{2},$$

then $|b(a,\gamma) - b(a,\gamma_0)| < \hat{\delta}$ with $\gamma \in [a_0 - \mu, a_0 + \mu]$. For given $\delta = 2\hat{\delta}$, we get the result. $\square$

4.5.2 $f < 0$ case for $\eta > a$

Now we consider f satisfying the equation

$$-\frac{1}{2}\eta f'(\eta) = [\varepsilon\phi'(-f(\eta))f'(\eta)]', \quad \eta > a. \tag{4.45}$$

At the boundaries we require

$$\lim_{\eta\to\infty} f(\eta) = V_0, \tag{4.46}$$

$$\lim_{\eta\searrow a} f(\eta) = 0, \quad \lim_{\eta\searrow a} \varepsilon\phi'(-f(\eta))f'(\eta) = -\gamma, \tag{4.47}$$

where a and $\gamma > 0$ are constants.

Following from what we studied on the positive solution, we can directly obtain the local existence, uniqueness results and continuity forward to $\eta = \infty$ of the solution f by the change of variables (4.35). As for $\eta < a$, we know that f is a monotonically decreasing function. Moreover, we know from Lemma 3.7 directly that $\lim_{\eta\to\infty} f'(\eta) = 0$. Similarly, if we define

$$d(a,\gamma) := \lim_{\eta\to\infty} f(\eta; a, \gamma),$$

the following properties of $d(a,\gamma)$ are obtained immediately using the change of variables (4.35).

Lemma 4.47. *$d(a,\gamma)$ has the following properties with fixed γ:*

(i) *$d(a,\gamma)$ is strictly monotonically increasing in a;*

(ii) $\lim_{a\to\infty} d(a,\gamma) = 0$;

(iii) *$d(a,\gamma)$ is a continuous function in a ;*

(iv) $\lim_{a\to-\infty} d(a,\gamma) = -\infty$.

Lemma 4.48. *$d(a,\gamma)$ has the following properties with fixed a:*

(i) *$d(a,\gamma)$ is strictly monotonically decreasing in γ;*

(ii) *$d(a,\gamma)$ is a continuous function in γ.*

(iii) $\lim\limits_{\gamma\to\infty} d(a,\gamma) = -\infty$.

(iv) $\lim\limits_{\gamma\to 0} d(a,\gamma) = 0$.

Lemma 4.49. *$d(a,\gamma)$ is a continuous function of γ and a.*

4.5.3 Two-parameter shooting method

Similarly to the half-line case, we will use a two-parameter shooting method to prove the existence of a self-similar solution of problem (4.29).

Theorem 4.50. *Suppose $\varepsilon > 0$, then there exists a unique solution f of problem (4.29).*

Proof. First we identify four "bad" sets

$$\begin{aligned}
\Gamma_1 &= \left\{(a,\gamma)\,\middle|\, b(a,\gamma) > U_0\right\},\\
\Gamma_2 &= \left\{(a,\gamma)\,\middle|\, b(a,\gamma) < U_0\right\},\\
\Gamma_3 &= \left\{(a,\gamma)\,\middle|\, d(a,\gamma) > -V_0\right\},\\
\Gamma_4 &= \left\{(a,\gamma)\,\middle|\, d(a,\gamma) < -V_0\right\}.
\end{aligned}$$

Now we combine the Γ_i to form two new sets: $\Lambda_1 = \Gamma_1 \cup \Gamma_4$, $\Lambda_2 = \Gamma_2 \cup \Gamma_3$. We show that Λ_1 and Λ_2 satisfy the hypothesis of Lemma 3.29 in the same way as half-line case, replacing $\lim\limits_{\eta\to 0} f(\eta; a,\gamma)$ with $\lim\limits_{\eta\to-\infty} f(\eta; a,\gamma)$. We will apply Lemma 3.29 to the set $\mathbb{R}\times(0,\infty)$, which is homeomorphic to the entire plane, for example, if we define a homeomorphism $g : \mathbb{R}\times(0,\infty) \mapsto \mathbb{R}^2$ such that $g(x,y) = (x, \log y)$.

These sets are clearly open in $(0,\infty)\times(0,\infty)$ since $b(a,\gamma)$ and $d(a,\gamma)$ are continuous function of a and γ by Lemma 4.46 and 4.49. Moreover, Lemma 4.44 and Lemma 4.47 yield that $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ are non-empty. Therefore, Λ_1 and Λ_2 are open and non-empty.

By using a similar argument to that in the proof of Lemma 3.31 in the half-line case, together with the monotonicity of $b(a,\gamma)$ and $d(a,\gamma)$, we can prove that $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ are each connected and $\Gamma_1\cap\Gamma_4\neq\emptyset$, $\Gamma_2\cap\Gamma_3\neq\emptyset$. Then, we can find $(a^*,\gamma^*)\in\Gamma_1\cap\Gamma_4$ that connects $(\hat{a},\hat{\gamma})\in\Gamma_1$ and $(\tilde{a},\tilde{\gamma})\in\Gamma_4$ since Γ_1,Γ_4 are each connected. Therefore, Λ_1 is connected, and similarly, Λ_2 is connected.

Next we will show that $\Lambda_1\cap\Lambda_2$ is disconnected. We have

$$\Lambda_1\cap\Lambda_2=(\Gamma_1\cap\Gamma_2)\cup(\Gamma_1\cap\Gamma_3)\cup(\Gamma_2\cap\Gamma_4)\cup(\Gamma_3\cap\Gamma_4).$$

Clearly $\Gamma_1\cap\Gamma_2$, $\Gamma_3\cap\Gamma_4$ are empty.

By using the similar argument as in the proof of Lemma 3.32, together with Lemma 4.44, 4.45, 4.47 and 4.48, we can prove that $\Gamma_1\cap\Gamma_3$ and $\Gamma_2\cap\Gamma_4$ are non-empty and disjoint. Therefore, $\Lambda_1\cap\Lambda_2$ is disconnected since it is union of non-empty, disjoint and open sets.

Then Lemma 3.29 yields that there is a point $(\bar{a},\bar{\gamma})\in\mathbb{R}\times(0,\infty)$ which is not in Λ_1 and Λ_2. In conclusion, $b(\bar{a},\bar{\gamma})=U_0$ since $(\bar{a},\bar{\gamma})\notin\Gamma_1\cup\Gamma_2$, $d(\bar{a},\bar{\gamma})=-V_0$ since $(\bar{a},\bar{\gamma})\notin\Gamma_3\cup\Gamma_4$. The result then follows from Theorems 4.28 and 4.31. $\square$

4.6 Self-similar solutions for $\varepsilon=0$

Now we consider

$$-\frac{1}{2}\eta f'(\eta)=[\phi'(f(\eta))f'(\eta)]',\quad \eta<a, \tag{4.48}$$

with boundaries required

$$
\begin{aligned}
&\lim_{\eta \to -\infty} f(\eta) = U_0, \\
&\lim_{\eta \nearrow a} f(\eta) = 0, \\
&\lim_{\eta \nearrow a} \phi'(f(\eta)) f'(\eta) = -\frac{aV_0}{2}.
\end{aligned}
\tag{4.49}
$$

For $\varepsilon = 0$ case, we know that $f(\eta) = -V_0$ for $\eta > a$ and $\gamma = \frac{aV_0}{2}$ which are the same as in the half-line case. Note that when $\varepsilon = 0$, a, γ are positive by Lemma 4.35. Since we showed that for each $a \in \mathbb{R}, \gamma > 0$, there exists solution f for $\eta \in (a - \delta, a)$ for some $\delta > 0$, then there exists a solution of (4.48) on interval $(a - \delta, \alpha)$. By Lemma 4.40 we know that with $\gamma = \frac{aV_0}{2}$ the solution f can be continued back to $-\infty$. Same as for the $\varepsilon > 0$ case, we know that f is a monotonically decreasing function.

Now define

$$
b(a) \colon = \lim_{\eta \to -\infty} f\left(\eta; a, \frac{aV_0}{2}\right).
$$

Note that we use the same notation $b(a)$ as in the half-line case, but here $b(a)$ define as the function of $f(\eta; a)$ as $\eta \to -\infty$ rather than $\eta \to 0$.

In order to prove the existence of similarity solution by using the parameter shooting, we first deduce the following properties of $b(a)$.

Lemma 4.51. *$b(a)$ has the following properties:*

(i) *$b(a)$ is strictly monotonically increasing in a;*

(ii) $\lim_{a \to 0} b(a) = 0$;

(iii) *$b(a)$ is a continuous function of a;*

(iv) $\lim_{a \to \infty} b(a) = \infty$;

Proof. We obtain (i), (iii) and (iv) directly from Lemma 3.33 and 4.44 replacing γ with $\frac{aV_0}{2}$, note that we can do that since γ is positive and is increasing in a. Now we want to prove $\lim_{a\to 0} b(a) = 0$ as $a > 0$.

Let $a < 1$, denote $N = \phi'(b(1))$, we have $\phi'(f) \leq N$ by (i). Then we get directly from the equation of η for $\eta < 0$ that

$$-\frac{\eta}{2N}[\phi(f(\eta))]' \geq [\phi(f(\eta))]'',$$

then we multiplying the inequality by $e^{\frac{-\eta^2}{4N}}$ and integrating from η to 0 yields

$$[\phi(f(\eta))]' \geq Ae^{\frac{-\eta^2}{4N}},$$

where $A = \phi'(f(0))f'(0) < 0$. Integrating again from $-\infty$ to 0 we get

$$\phi(b(a)) \leq \phi(f(0)) - A\int_{-\infty}^{0} e^{\frac{-s^2}{4N}}\mathrm{d}s.$$

Integrating the equation (4.48) from 0 to a yields

$$\frac{1}{2}\int_0^a f(s)\mathrm{d}s = \frac{aV_0}{2} - \phi'(f(0))f'(0).$$

Then we have

$$-A = \frac{aV_0}{2} + \frac{1}{2}\int_0^a f(s)\mathrm{d}s \to 0 \text{ as } a \to 0.$$

Therefore $\phi(b(a)) \leq \phi(f(0)) - A\int_{-\infty}^{0} e^{\frac{-s^2}{4N}}\mathrm{d}s \to 0$ as $a \to 0$, since $\int_{-\infty}^{0} e^{\frac{-s^2}{4N}} < \infty$ and $\phi(f(0)) \to 0$ as $a \to 0$ by Lemma 3.18 (ii) and $\phi(0) = 0$. $\square$

Similarly to Theorem 3.34, we now prove that for some $a > 0$, there exists a solution satisfying (4.48) by using a one-parameter shooting method.

Theorem 4.52. *Suppose $\varepsilon = 0$, then there exists a unique solution f of problem (4.30).*

Proof. Identify two "bad" sets

$$S^- = \left\{ a \,\middle|\, b(a) < U_0 \right\},$$
$$S^+ = \left\{ a \,\middle|\, b(a) > U_0 \right\}.$$

Clearly S^- and S^+ are disjoint. By Lemma 4.51 (ii) and (iv), we know that S^- and S^+ are non-empty. Moreover, S^- and S^+ are open. Indeed, let $a_0 \in S^-$ and let $\beta := U_0 - b(a_0)$, then there exists μ such that $|b(a) - b(a_0)| < \beta$ for $|a - a_0| < \mu$, since $b(a)$ is continuous by Lemma 3.33, which implies $b(a) < b(a_0) + \beta < U_0$. A similar proof show that S^+ is open. Since S^- and S^+ are non-empty disjoint open sets, $S^- \cup S^+ \neq \mathbb{R}$. Then we can conclude that there exists $\bar{a} \notin S^- \cup S^+$, which yields $b(\bar{a}) = U_0$. $\square$

Chapter 5

Self-similar solutions with special $\phi'(f) = f^{m-1}$ with $m > 1$

The choice of ϕ satisfying (1.2) and (1.3) plays an important role in the characterisation of rates at which one substance invades another of the system (1.6). For concreteness, we consider the specific family that is motivated by porous medium equation

$$\phi'(w) = w^{m-1} \tag{5.1}$$

with $m > 1$, which satisfies the conditions (1.2) and (1.3).

The form of self-similar solution of the limit problems with nonlinear diffusion $w(x,t) = f(\eta)$ is exactly same as in the linear diffusion case where $\eta = \frac{x}{\sqrt{t}}$ is independent of the choice of ϕ. We are interested in how the free boundary is affected by m, in the other words, the relationship between m and a, where a gives the position of free-boundary because $f(a) = 0$. In the following section, we will explore the self-similar solution $f_m(\eta) = f(\eta; m)$, in particular, how the value a depends on m.

5.1 The whole-line case with $\varepsilon \geq 0$

First we consider the whole line case with the specific choice of ϕ'. For $\varepsilon > 0$, the problem satisfied by f is

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]', & \text{if } -\infty < \eta < a, \\ -\dfrac{1}{2}\eta f'(\eta) = \varepsilon[(-f)^{m-1}(\eta)f'(\eta)]', & \text{if } a < \eta < \infty, \\ \lim\limits_{\eta \to -\infty} f(\eta) = U_0, \quad \lim\limits_{\eta \to \infty} f(\eta) = -V_0, \\ \lim\limits_{\eta \nearrow a} f(\eta) = 0 = -\lim\limits_{\eta \searrow a} f(\eta), \\ \lim\limits_{\eta \nearrow a} f^{m-1}(\eta)f'(\eta) = \varepsilon \lim\limits_{\eta \searrow a}(-f)^{m-1}(\eta)f'(\eta), \end{cases} \tag{5.2}$$

whereas for $\varepsilon = 0$, the problem satisfied by f is

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]', & \text{if } -\infty < \eta < a, \\ f(\eta) = -V_0, & \text{if } a < \eta < \infty, \\ \lim\limits_{\eta \to -\infty} f(\eta) = U_0, \\ \lim\limits_{\eta \nearrow a} f(\eta) = 0, \\ \lim\limits_{\eta \nearrow a} f^{m-1}(\eta)f'(\eta) = -\dfrac{aV_0}{2}, \end{cases} \tag{5.3}$$

where a is positive.

Recall $f_{m_i}(\eta) = f(\eta; m_i)$, denote a_{m_i} be the position of free boundary where $f_{m_i}(a_{m_i}) = 0$, and $\gamma_{m_i} = -\lim\limits_{\eta \nearrow a_{m_i}} f_{m_i}^{m_i-1}(\eta)f'_{m_i}(\eta)$.

Consider f_{m_1} and f_{m_2} satisfy (5.2) with $m_1 \neq m_2$, we will deduce some results about intersection of f_{m_1} and f_{m_2}.

Lemma 5.1. *Suppose $a_{m_1} < a_{m_2}$, if f_{m_1} and f_{m_2} satisfy (5.2), then we have*

(i) *for $\varepsilon > 0$, there exists some $\eta_0 \in \mathbb{R}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$;*

(ii) *for $\varepsilon = 0$, there exists some $\eta_0 < a_{m_1}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$.*

Proof. (i) For $\varepsilon > 0$, suppose there exists no η_0 such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$. Then we must have $f_{m_1} < f_{m_2}$ for all $\eta \in \mathbb{R}$ since $a_{m_1} < a_{m_2}$.

We need to consider both equations that are satisfied by f for a given m, depending on whether $f > 0$ or $f < 0$

$$-\frac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]', \qquad \text{if } \eta < a, \tag{5.4}$$

$$-\frac{1}{2}\eta f'(\eta) = \varepsilon[(-f)^{m-1}(\eta)f'(\eta)]', \qquad \text{if } \eta > a, \tag{5.5}$$

If f_{m_1} and f_{m_2} are solutions of (5.4) corresponding to m_1 and m_2, then we integrate (5.4) from η to a_{m_1}, a_{m_2} and obtain

$$-\frac{1}{2}\eta f_{m_1}(\eta) + \frac{1}{2}\int_{\eta}^{a_{m_1}} f_{m_1}(s)\mathrm{d}s = -\gamma_{m_1} - f_{m_1}^{m_1-1}(\eta)f'_{m_1}(\eta), \tag{5.6}$$

$$-\frac{1}{2}\eta f_{m_2}(\eta) + \frac{1}{2}\int_{\eta}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s = -\gamma_{m_2} - f_{m_2}^{m_2-1}(\eta)f'_{m_2}(\eta). \tag{5.7}$$

Subtracting (5.6) from (5.7) and letting $\eta \to \infty$ we have

$$\lim_{\eta\to-\infty} \eta\left(f_{m_1}(\eta) - f_{m_2}(\eta)\right) + \frac{1}{2}\int_{-\infty}^{a_{m_1}} \left[f_{m_2}(s) - f_{m_1}(s)\right]\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s$$
$$= \gamma_{m_1} - \gamma_{m_2},$$

since $\lim_{\eta\to-\infty} f'(\eta) = 0$ by Lemma 4.43. We know that $\lim_{\eta\to-\infty} \eta\left(U_0 - f(\eta)\right) = 0$ by Corollary 4.30, since f converges to U_0 exponentially, which yields

$$\lim_{\eta\to-\infty} \eta\left(f_{m_1}(\eta) - f_{m_2}(\eta)\right) = -\lim_{\eta\to-\infty} \eta(U_0 - f_{m_2}(\eta)) + \lim_{\eta\to-\infty} \eta(U_0 - f_{m_1}(\eta)) \to 0.$$

Then we have

$$\frac{1}{2}\int_{-\infty}^{a_{m_1}} \left[f_{m_2}(s) - f_{m_1}(s)\right]\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s = \gamma_{m_1} - \gamma_{m_2}. \tag{5.8}$$

We know that the left-hand side of (5.8) is positive since $f_{m_1} < f_{m_2}$ for $\eta < a_{m_1}$.

Similarly, integrating (5.5) from a_{m_1}, a_{m_2} to η, subtracting the two equations and letting $\eta \to -\infty$ yield

$$\frac{1}{2}\int_{a_{m_2}}^{\infty} [f_{m_1}(s) - f_{m_2}(s)]\,\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_1}(s)\mathrm{d}s = \gamma_{m_1} - \gamma_{m_2}. \tag{5.9}$$

The left-hand side of (5.9) is negative since $f_{m_1} < f_{m_2}$ for $\eta > a_{m_2}$.

Note that for given m, a_m and γ_m are uniquely determined by U_0 and V_0. If $a_{m_1} < a_{m_2}$, we need to consider all possible ordering of $\gamma_{m_1}, \gamma_{m_2}$, since when $\varepsilon > 0$, it is not clear whether $\gamma_{m_1} > \gamma_{m_2}$, $\gamma_{m_1} < \gamma_{m_2}$ or $\gamma_{m_1} = \gamma_{m_2}$.

In conclusion, for $a_{m_1} < a_{m_2}$ we have

(a) if $\gamma_{m_1} < \gamma_{m_2}$, the right-hand side of (5.8) is negative, then we obtain a contradiction and therefore there must exist some $\eta_0 < a_{m_1}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$;

(b) if $\gamma_{m_1} > \gamma_{m_2}$, the right-hand side of (5.9) is positive, then we obtain a contradiction and therefore there must exist some $\eta_0 > a_{m_2}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$.

(c) if $\gamma_{m_1} = \gamma_{m_2}$, the right-hand sides of (5.8) and (5.9) are both 0, then we obtain a contradiction with both (5.8) and (5.9) and therefore there must exist some $\eta_1 < a_{m_1}$ and $\eta_2 > a_{m_2}$ such that $f_{m_1}(\eta_i) = f_{m_2}(\eta_i)$ where $i = 1, 2$.

(ii) We use a similar proof as in (i). For $\varepsilon = 0$, suppose there exists no $\eta_0 < a_{m_1}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$. Then we must have $f_{m_1} < f_{m_2}$ for all $\eta \in \mathbb{R}$ since $a_{m_1} < a_{m_2}$.

We consider

$$-\frac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta) f'(\eta)]', \quad \eta < a, \tag{5.10}$$

with $\gamma = \frac{aV_0}{2}$. If f_{m_1} and f_{m_2} are solution of (5.10) with corresponding m_1, m_2, then, integrating the equation of f from η to a_{m_1}, a_{m_2}, subtracting the equations and letting $\eta \to -\infty$ yields

$$\frac{1}{2}\int_{-\infty}^{a_{m_1}} [f_{m_2}(s) - f_{m_1}(s)]\,\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s = \frac{a_{m_1}V_0}{2} - \frac{a_{m_2}V_0}{2}.$$

We know that the left-hand side is positive since $f_{m_2} > f_{m_1}$ for $\eta < a_{m_1}$. For $a_{m_1} < a_{m_2}$, the left-hand side is negative, then there is a contradiction, then there must exists $\eta_0 < a_{m_1}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$. $\square$

5.1.1 Results in the special case $U_0, V_0 < 1$ and $m \geq 2$

The following results will proved by using contradiction arguments that rely on $|f(\eta)| < 1$, which is ensured by imposing the additional conditions that $U_0, V_0 < 1$ and $m \geq 2$.

First, we consider all possible orderings of $\gamma_{m_1}, \gamma_{m_2}$ when $a_{m_1} < a_{m_2}$, namely $\gamma_{m_1} > \gamma_{m_2}$, $\gamma_{m_1} < \gamma_{m_2}$, or $\gamma_{m_1} = \gamma_{m_2}$, and then prove the relationship between a and m in each case.

Lemma 5.2. *Let f_{m_1}, f_{m_2} satisfy (5.2) with corresponding $m_1, m_2 \geq 2$ and $U_0, V_0 < 1$. Suppose $a_{m_1} < a_{m_2}$. Then we have*

(i) $\gamma_{m_1} \neq \gamma_{m_2}$*;*

(ii) if $\gamma_{m_1} < \gamma_{m_2}$*, then* $m_1 > m_2$*;*

(iii) if $\gamma_{m_1} > \gamma_{m_2}$*, then* $m_1 < m_2$*.*

Proof. We know from the proof of Lemma 5.1 that in the whole line case, for $a_{m_1} < a_{m_2}$, then

(A) if $\gamma_{m_1} < \gamma_{m_2}$, then there must exist some $\eta_0 < a_{m_1}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$;

(B) if $\gamma_{m_1} > \gamma_{m_2}$, then there must exist some $\eta_0 > a_{m_2}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$;

(C) if $\gamma_{m_1} = \gamma_{m_2}$, then there must exist some $\eta_1 < a_{m_1}$ and $\eta_2 > a_{m_2}$ such that $f_{m_1}(\eta_i) = f_{m_2}(\eta_i)$, where $i = 1, 2$.

The results are proved by considering different cases as above. We begin by proving (ii) and (iii) by looking at (A) and (B).

(A) Suppose $a_{m_1} < a_{m_2}$ and $\gamma_{m_1} < \gamma_{m_1}$, then there exists $\eta_1 < a_{m_1}$ closest to a_{m_1} such that $f_{m_1}(\eta_1) = f_{m_2}(\eta_1)$. Integrating (5.4) from η_1 to a_{m_1} and a_{m_2} we get

$$-\frac{1}{2}\eta_1 f_{m_1}(\eta_1) + \frac{1}{2}\int_{\eta_1}^{a_{m_1}} f_{m_1}(s)\mathrm{d}s = -\gamma_{m_1} - f_{m_1}^{m_1-1}(\eta_1) f'_{m_1}(\eta_1), \tag{5.11}$$

$$-\frac{1}{2}\eta_1 f_{m_2}(\eta_1) + \frac{1}{2}\int_{\eta_1}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s = -\gamma_{m_2} - f_{m_2}^{m_2-1}(\eta_1) f'_{m_2}(\eta_1). \tag{5.12}$$

Subtracting (5.11) from (5.12) we have

$$\begin{aligned}&\frac{1}{2}\int_{\eta_1}^{a_{m_1}} \left[f_{m_2}(s) - f_{m_1}(s)\right] \mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s + \gamma_{m_2} - \gamma_{m_1} \\ =& f_{m_1}^{m_1-1}(\eta_1) f'_{m_1}(\eta_1) - f_{m_2}^{m_2-1}(\eta_1) f'_{m_2}(\eta_1).\end{aligned} \tag{5.13}$$

The left-hand side of (5.13) is positive since $f_{m_2} > f_{m_1}$ for $\eta \in (\eta_1, a_{m_1})$, then we get

$$f_{m_1}^{m_1-1}(\eta_1) f'_{m_1}(\eta_1) - f_{m_2}^{m_2-1}(\eta_1) f'_{m_2}(\eta_1) > 0.$$

The fact that $-f'_{m_1}(\eta_1) > -f'_{m_2}(\eta_1) > 0$ gives

$$f_{m_1}^{m_1-1}(\eta_1) < f_{m_2}^{m_2-1}(\eta_1).$$

Therefore, if $2 \leq m_1 \leq m_2$, we have $f_{m_1}(\eta_1) = f_{m_2}(\eta_1) \geq 1$ which is impossible since $U_0 < 1$ and f is decreasing, this proves (ii). Note that if $m_1 > m_2$, we have $f_{m_1}(\eta_1) = f_{m_2}(\eta_1) < 1$, which does not contradict $U_0 < 1$.

(B) Suppose $a_{m_1} < a_{m_2}$ and $\gamma_{m_1} > \gamma_{m_1}$, then there exists $\eta_2 > a_{m_2}$ closest to a_{m_2} such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_2)$.

Integrating (5.5) from a_{m_1} and a_{m_2} to η_2 we get

$$-\frac{1}{2}\eta_2 f_{m_1}(\eta_2) + \frac{1}{2}\int_{a_{m_1}}^{\eta_2} f_{m_1}(s)\mathrm{d}s = \gamma_{m_1} + \varepsilon\big(-f_{m_1}\big)^{m_1-1}(\eta_2)f'_{m_1}(\eta_2), \quad (5.14)$$

$$-\frac{1}{2}\eta_2 f_{m_2}(\eta_2) + \frac{1}{2}\int_{a_{m_2}}^{\eta_2} f_{m_2}(s)\mathrm{d}s = \gamma_{m_2} + \varepsilon\big(-f_{m_2}\big)^{m_2-1}(\eta_2)f'_{m_2}(\eta_2). \quad (5.15)$$

Subtracting (5.15) from (5.14) we have

$$\begin{aligned}&\frac{1}{2}\int_{a_{m_2}}^{\eta_2}\left[f_{m_1}(s) - f_{m_2}(s)\right]\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_1}(s)\mathrm{d}s + \gamma_{m_2} - \gamma_{m_1}\\ =&\varepsilon\big(-f_{m_1}\big)^{m_1-1}(\eta_2)f'_{m_1}(\eta_2) - \varepsilon\big(-f_{m_2}\big)^{m_2-1}(\eta_2)f'_{m_2}(\eta_2). \qquad (5.16)\end{aligned}$$

The left-hand side of (5.16) is negative since $f_{m_2} > f_{m_1}$ for $\eta \in (a_{m_2}, \eta_2)$, then we get

$$\big(-f_{m_1}\big)^{m_1-1}(\eta_2)f'_{m_1}(\eta_2) - \big(-f_{m_2}\big)^{m_2-1}(\eta_2)f'_{m_2}(\eta_2) < 0.$$

The fact that $0 < -f'_{m_1}(\eta_2) < -f'_{m_2}(\eta_2)$ gives

$$\big(-f_{m_1}\big)^{m_1-1}(\eta_2) > \big(-f_{m_2}\big)^{m_2-1}(\eta_2).$$

Therefore, if $2 \leq m_2 \leq m_1$, we have $f_{m_1}(\eta_2) = f_{m_2}(\eta_2) \leq -1$ which is impossible since $-V_0 > -1$ and f is decreasing, this proves (iii). Note that if $m_2 > m_1$, we have $f_{m_1}(\eta_2) = f_{m_2}(\eta_2) > -1$, which does not contradict $V_0 < 1$.

(C) Suppose $a_{m_1} < a_{m_2}$ and $\gamma_{m_1} = \gamma_{m_2}$, then there exist $\eta_1 < a_{m_1}$ closest to

a_{m_1} and $\eta_2 > a_{m_2}$ closest to a_{m_2} such that $f_{m_1}(\eta_i) = f_{m_2}(\eta_i)$ where $i = 1, 2$. Similar as in (A), if we integrate (5.4) from η_1 to a_{m_1}, a_{m_2} and subtract two equations we get

$$\frac{1}{2}\int_{\eta_1}^{a_{m_1}} [f_{m_2}(s) - f_{m_1}(s)]\,\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s \tag{5.17}$$

$$= f_{m_1}^{m_1-1}(\eta_1)f'_{m_1}(\eta_1) - f_{m_2}^{m_2-1}(\eta_1)f'_{m_2}(\eta_1). \tag{5.18}$$

Since the left-hand side of (5.18) is positive, we have $f_{m_1}^{m_1-1}(\eta_1) < f_{m_2}^{m_2-1}(\eta_1)$, yields $m_2 < m_1$. The estimate for η_2 is the same as in (B), we have

$$\frac{1}{2}\int_{a_{m_2}}^{\eta_2} [f_{m_1}(s) - f_{m_2}(s)]\,\mathrm{d}s + \int_{a_{m_1}}^{a_{m_2}} f_{m_1}(s)\mathrm{d}s$$

$$= \varepsilon\big(-f_{m_1}\big)^{m_1-1}(\eta_2)f'_{m_1}(\eta_2) - \varepsilon\big(-f_{m_2}\big)^{m_2-1}(\eta_2)f'_{m_2}(\eta_2). \tag{5.19}$$

We have $m_2 > m_1$ since the left-hand side of (5.19) is negative. There is a contradiction, so it is impossible that $\gamma_1 = \gamma_2$ as $a_{m_1} < a_{m_1}$. □

Next we will study how the behaviour of f close to $-\infty$ and ∞ depends on m by looking at the point η_0 closest to $-\infty$ and ∞ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$.

Lemma 5.3. *Let f_{m_1}, f_{m_2} satisfy (5.2) with corresponding $m_1, m_2 \geq 2$ and $U_0, V_0 < 1$. Assuming there do not exist sequences of intersection points of f_{m_1}, f_{m_2} tending to $-\infty, \infty$ and suppose $a_{m_1} < a_{m_2}$, then we have*

(i) if $\gamma_{m_1} < \gamma_{m_2}$, then $m_1 > m_2$ and there exist $\hat{\eta} < 0$ such that $f_{m_1}(\eta) > f_{m_2}(\eta)$ for $\eta < \hat{\eta}$;

(ii) if $\gamma_{m_1} > \gamma_{m_2}$, then $m_1 < m_2$ and there exist $\tilde{\eta} > 0$ such that $f_{m_1}(\eta) > f_{m_2}(\eta)$ for $\eta > \tilde{\eta}$.

Proof. (i) First note that by Lemma 5.2, $m_1 > m_2$ and the proof of Lemma 5.1, there exists $\hat{\eta} < a_{m_1}$ such that $f_{m_1}(\hat{\eta}) = f_{m_2}(\hat{\eta})$. Let $\hat{\eta} < a_{m_1}$ be the intersection point of f_{m_1}, f_{m_2} closest to $-\infty$, that is $f_{m_1}(\hat{\eta}) = f_{m_2}(\hat{\eta})$ and $f_{m_1}(\eta) \neq f_{m_2}(\eta)$ for $\eta < \hat{\eta}$.

Integrating (5.4) from η to $\hat{\eta}$ we get

$$\begin{aligned}
&-\frac{1}{2}\hat{\eta} f_{m_1}(\hat{\eta}) + \frac{1}{2}\eta f_{m_1}(\eta) + \frac{1}{2}\int_{\eta}^{\hat{\eta}} f_{m_1}(s)\mathrm{d}s \\
=&\big(f_{m_1}\big)^{m_1-1}(\hat{\eta}) f'_{m_1}(\hat{\eta}) - \big(f_{m_1}\big)^{m_1-1}(\eta) f'_{m_1}(\eta), \qquad (5.20)\\
&-\frac{1}{2}\hat{\eta} f_{m_2}(\hat{\eta}) + \frac{1}{2}\eta f_{m_2}(\eta) + \frac{1}{2}\int_{\eta}^{\hat{\eta}} f_{m_2}(s)\mathrm{d}s \\
=&\big(f_{m_2}\big)^{m_2-1}(\hat{\eta}) f'_{m_2}(\hat{\eta}) - \big(f_{m_2}\big)^{m_2-1}(\eta) f'_{m_2}(\eta). \qquad (5.21)
\end{aligned}$$

Subtracting (5.21) from (5.20) and letting $\eta \to -\infty$ we have

$$\frac{1}{2}\int_{-\infty}^{\eta_3} \left[f_{m_1}(s) - f_{m_2}(s)\right]\mathrm{d}s = \big(f_{m_1}\big)^{m_1-1}(\eta_3) f'_{m_1}(\eta_3) - \big(f_{m_2}\big)^{m_2-1}(\eta_3) f'_{m_2}(\eta_3), \qquad (5.22)$$

by Lemma 4.43 and Corollary 4.30, f converges to U_0 exponentially, which yields

$$\begin{aligned}
&\lim_{\eta_3\to-\infty} \eta_3\big(f_{m_1}(\eta_3) - f_{m_2}(\eta_3)\big) \\
=&-\lim_{\eta_3\to-\infty} \eta_3\big(U_0 - f_{m_2}(\eta_3)\big) + \lim_{\eta_3\to-\infty} \eta_3\big(U_0 - f_{m_1}(\eta_3)\big) \to 0.
\end{aligned}$$

We have $\big(f_{m_1}\big)^{m_1-1}(\hat{\eta}) < \big(f_{m_2}\big)^{m_2-1}(\hat{\eta})$, since $U_0 < 1$ and f is decreasing, so $f_{m_1}(\hat{\eta}) = f_{m_2}(\hat{\eta}) < 1$ and $m_1 > m_2$. Suppose that $f_{m_1}(\eta) \leq f_{m_2}(\eta)$ for $\eta < \hat{\eta}$, then the right-hand side of (5.22) is positive since $-f'_{m_1}(\eta_3) \leq -f'_{m_2}(\eta_3)$, which contradicts $\frac{1}{2}\int_{-\infty}^{\tilde{\eta}} \left[f_{m_1}(s) - f_{m_2}(s)\right]\mathrm{d}s \leq 0$. Therefore, $f_{m_1}(\eta) > f_{m_2}(\eta)$ for $\eta < \hat{\eta}$.

(ii) First note that by Lemma 5.2, $m_2 > m_1$, and by the proof of Lemma 5.1, there exists $\tilde{\eta} > a_{m_2}$ such that $f_{m_1}(\tilde{\eta}) = f_{m_2}(\tilde{\eta})$. Let $\tilde{\eta} > a_{m_2}$ be the intersection point of f_{m_1}, f_{m_2} closest to ∞, that is $f_{m_1}(\tilde{\eta}) = f_{m_2}(\tilde{\eta})$ and $f_{m_1}(\eta) \neq f_{m_2}(\eta)$ for $\eta > \tilde{\eta}$.

Integrating (5.5) from $\tilde{\eta}$ to η we get

$$\begin{aligned}
&-\frac{1}{2}\eta f_{m_1}(\eta) + \frac{1}{2}\tilde{\eta} f_{m_1}(\tilde{\eta}) + \frac{1}{2}\int_{\tilde{\eta}}^{\eta} f_{m_1}(s)\mathrm{d}s \\
=&\varepsilon\big(-f_{m_1}\big)^{m_1-1}(\eta) f'_{m_1}(\eta) - \varepsilon\big(-f_{m_1}\big)^{m_1-1}(\tilde{\eta}) f'_{m_1}(\tilde{\eta}), \qquad (5.23)\\
&-\frac{1}{2}\eta f_{m_2}(\eta) + \frac{1}{2}\tilde{\eta} f_{m_2}(\tilde{\eta}) + \frac{1}{2}\int_{\tilde{\eta}}^{\eta} f_{m_2}(s)\mathrm{d}s \\
=&\varepsilon\big(-f_{m_2}\big)^{m_2-1}(\eta) f'_{m_2}(\eta) - \varepsilon\big(-f_{m_2}\big)^{m_2-1}(\tilde{\eta}) f'_{m_2}(\tilde{\eta}). \qquad (5.24)
\end{aligned}$$

Subtracting (5.23) from (5.24) and letting $\eta \to \infty$ we have

$$\begin{aligned}
&\frac{1}{2}\int_{\eta_4}^{\infty} \left[f_{m_2}(s) - f_{m_1}(s)\right] \mathrm{d}s \\
=&\varepsilon\big(-f_{m_1}\big)^{m_1-1}(\eta_4) f'_{m_1}(\eta_4) - \varepsilon\big(-f_{m_2}\big)^{m_2-1}(\eta_4) f'_{m_2}(\eta_4), \qquad (5.25)
\end{aligned}$$

by Lemma 4.43 and Corollary 4.30, f converges to $-V_0$ exponentially, which yields

$$\begin{aligned}
&\lim_{\eta_4\to\infty} \eta_4\left(f_{m_1}(\eta_4) - f_{m_2}(\eta_4)\right) \\
=&\lim_{\eta_4\to\infty} \eta_4(f_{m_1}(\eta_4) + V_0) - \lim_{\eta_4\to\infty} \eta_4(f_{m_2}(\eta_4) + V_0) \to 0.
\end{aligned}$$

We have $(-f_{m_1})^{m_1-1}(\tilde{\eta}) > (-f_{m_2})^{m_2-1}(\tilde{\eta})$, since $V_0 < 1$ and f is decreasing, so $(-f_{m_1})(\tilde{\eta}) = (-f_{m_2})(\tilde{\eta}) < 1$ and $m_2 > m_1$. Suppose that $f_{m_1}(\eta) \leq f_{m_2}(\eta)$ for $\eta > \tilde{\eta}$, then the right-hand side of (5.25) is negative since $-f'_{m_1}(\tilde{\eta}) \geq -f'_{m_2}(\tilde{\eta})$, which contradicts $\frac{1}{2}\int_{\tilde{\eta}}^{\infty}\left[f_{m_2}(s) - f_{m_1}(s)\right]\mathrm{d}s \geq 0$. Therefore, $f_{m_1}(\eta) > f_{m_2}(\eta)$ for $\eta > \tilde{\eta}$. $\square$

We can obtain the following results when $\varepsilon = 0$ by using the fact that $\gamma = \frac{aV_0}{2}$. In the following result, we only study the positive solutions $f(\eta)$ for $\eta < a$, so we consider $0 < U_0 < 1$ and $V_0 > 0$. Note that the relationship between a and m tells us how the speed of one substance penetrating into the other is affected by m.

Theorem 5.4. *Let $\varepsilon = 0$ and $U_0 < 1$, suppose f_{m_1}, f_{m_2} satisfy (5.3) with corresponding $m_1, m_2 \geq 2$ and a_{m_1}, a_{m_2}. Then if $m_1 > m_2$, we have*

$$0 < a_{m_1} < a_{m_2},$$

and assuming there does not exist a sequence of intersection points of f_{m_1}, f_{m_2} tending to $-\infty$, then there exists $\hat{\eta} < 0$ such that

$$f_{m_1}(\eta) < f_{m_2}(\eta) \quad \text{for} \quad \eta < \hat{\eta}.$$

Proof. For $\varepsilon = 0$ case, we have $\gamma = \frac{aV_0}{2} > 0$ which satisfies **(A)**, then by Lemma 5.1, there exists $\eta_0 < \min\{a_{m_1}, a_{m_2}\}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$. By a similar argument to Lemma 5.2 (i), we know that $a_{m_1} \neq a_{m_2}$, and $a_{m_1}, a_{m_2} > 0$ since $\gamma_{m_1}, \gamma_{m_2} > 0$.

Now let η_0 be the closet intersection point to $\min\{a_{m_1}, a_{m_2}\}$, integrating (5.3) from η_0 to a_{m_1}, a_{m_2} we get

$$-\frac{1}{2}\eta_0 f_{m_1}(\eta_0) + \frac{1}{2}\int_{\eta_0}^{a_{m_1}} f_{m_1}(s)\mathrm{d}s = -\frac{a_{m_1}V_0}{2} - f_{m_1}^{m_1-1}(\eta_0) f'_{m_1}(\eta_0), \qquad (5.26)$$

$$-\frac{1}{2}\eta_0 f_{m_2}(\eta_0) + \frac{1}{2}\int_{\eta_0}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s = -\frac{a_{m_2}V_0}{2} - f_{m_2}^{m_2-1}(\eta_0) f'_{m_2}(\eta_0). \qquad (5.27)$$

Subtracting (5.26) from (5.27) we have

$$\frac{1}{2}\int_{\eta_0}^{a_{m_1}} f_{m_1}(s)\mathrm{d}s - \frac{1}{2}\int_{\eta_0}^{a_{m_2}} f_{m_2}(s)\mathrm{d}s + \frac{a_{m_1}V_0}{2} - \frac{a_{m_2}V_0}{2}$$

$$=f_{m_1}^{m_1-1}(\eta_0)f'_{m_1}(\eta_0) - f_{m_2}^{m_2-1}(\eta_0)f'_{m_2}(\eta_0). \tag{5.28}$$

For $m_1 > m_2$ we know $f_{m_1}^{m_1-1}(\eta_0) < f_{m_2}^{m_2-1}(\eta_0)$, since $U_0 < 1$ and f is decreasing. Then if $a_{m_1} > a_{m_2}$, the left-hand side of (5.28) is positive and $-f'_{m_1}(\eta_0) < -f'_{m_2}(\eta_0)$, which gives

$$f_{m_1}^{m_1-1}(\eta_0)f'_{m_1}(\eta_0) - f_{m_2}^{m_2-1}(\eta_0)f'_{m_2}(\eta_0) < 0,$$

which contradicts the left-hand side is positive. Therefore if $m_1 > m_2$, we have $a_{m_1} < a_{m_2}$.

The result for the behaviour of f close to infinity follows from Lemma 5.3 (i). $\square$

5.2 The half-line case with $\varepsilon \geq 0$

Now we consider the half-line case with specific choice of ϕ'. For $\varepsilon > 0$, the problem satisfied by f is

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]', & \text{if } 0 < \eta < a, \\ -\dfrac{1}{2}\eta f'(\eta) = \varepsilon[(-f)^{m-1}(\eta)f'(\eta)]', & \text{if } a < \eta < \infty, \\ \lim\limits_{\eta\to 0} f(\eta) = U_0, \quad \lim\limits_{\eta\to\infty} f(\eta) = -V_0, & \\ \lim\limits_{\eta\nearrow a} f(\eta) = 0 = -\lim\limits_{\eta\searrow a} f(\eta), & \\ \lim\limits_{\eta\nearrow a} f^{m-1}(\eta)f'(\eta) = \varepsilon\lim\limits_{\eta\searrow a}(-f)^{m-1}(\eta)f'(\eta), & \end{cases} \tag{5.29}$$

whereas for $\varepsilon = 0$, the problem satisfied by f is

$$\begin{cases} -\dfrac{1}{2}\eta f'(\eta) = [f^{m-1}(\eta)f'(\eta)]', & \text{if } 0 < \eta < a, \\ f(\eta) = -V_0, & \text{if } a < \eta < \infty, \\ \lim\limits_{\eta \to 0} f(\eta) = U_0, \\ \lim\limits_{\eta \nearrow a} f(\eta) = 0, \\ \lim\limits_{\eta \nearrow a} f^{m-1}(\eta)f'(\eta) = \dfrac{aV_0}{2}. \end{cases} \tag{5.30}$$

Suppose f_{m_1}, f_{m_2} satisfy (5.29) and $a_{m_1} < a_{m_2}$, we know that $f_{m_1}(0) = f_{m_2}(0) = U_0$ and obtain directly from the proof of Lemma 5.1 that if $\gamma_{m_1} > \gamma_{m_2}$, there exists $\eta_0 > a_{m_2}$ such that $f_{m_1}(\eta_0) = f_{m_2}(\eta_0)$.

Now consider $U_0, V_0 < 1$, we can get the following Lemma by using a similar argument in the proof of Lemma 5.2. Note that it is not clear if there exists $\eta_1 \in (0, a_{m_1})$ such that $f_{m_1}(\eta_1) = f_{m_2}(\eta_1)$ but the key point is that we know that $f_{m_1}(0) = f_{m_2}(0)$, and this allows us to apply arguments as in the proof of Lemma 5.2, whether or not an additional intersection point $\eta_1 \in (0, a_{m_1})$ exists.

Lemma 5.5. *Let f_{m_1}, f_{m_2} satisfy (5.29) with corresponding $m_1, m_2 \geq 2$ and $U_0, V_0 < 1$. Suppose $a_{m_1} < a_{m_2}$, we have*

(i) $\gamma_{m_1} \neq \gamma_{m_2}$;

(ii) if $\gamma_{m_1} < \gamma_{m_2}$, then $m_1 > m_2$;

(iii) if $\gamma_{m_1} > \gamma_{m_2}$, then $m_1 < m_2$.

Suppose f_{m_1}, f_{m_2} satisfy (5.30) where $\varepsilon = 0$. Then $f_{m_1}(\eta) = f_{m_2}(\eta) = -V_0$ for $\eta > \max\{a_{m_1}, a_{m_2}\}$ and we know that $f_{m_1}(0) = f_{m_2}(0)$, thus the similar results to those in Theorem 5.4 can be obtained by using the same method as in the proof in whole-line case .

Theorem 5.6. *Let $\varepsilon = 0$ and $U_0 < 1$, and suppose f_{m_1}, f_{m_2} satisfy (5.30) with corresponding $m_1, m_2 \geq 2$ and a_{m_1}, a_{m_2}. Then*

$$\text{if} \quad m_1 > m_2, \quad \text{then} \quad a_{m_1} < a_{m_2}.$$

Bibliography

[1] R.A.Adams, J.J.F.Fournier, Sobolev Spaces, Second edition, Elsevier/Academic Press, Amsterdam, 2003.

[2] F.V.Atkinson, L.A.Peletier, Similarity profiles of flows through porous media, Arch. Rational Mech. Anal. 42(1971) 369-379.

[3] H.Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.

[4] M.Bertsch, R.Kersner, L.A.Peletier, Positivity versus localization in degenerate diffusion equations, Nonliear Anal. 9(9)(1985) 987-1008.

[5] J.E.Bouillet, S.M.Gomes, An equation with a singular nonlinearity related to diffusion problems in one dimension, Quart. Appl. Math. 42(4)(1985) 395-402.

[6] C.Chicone, Ordinary Differential Equations with Applications, Springer-Verlag, New York, 1999.

[7] A.H.Craven, L.A.Peletier, Similarity solutions for degenerate quasilinear parabolic equations, J. Math. Anal. Appl. 38(1972) 73-81.

[8] L.S.Leibenson, The Motion of a Gas in a Porous Medium, Russian Academu of Seciences, Moscow, 1930.

[9] E.C.M.Crooks, E.N.Dancer, D.Hilhorst, M.Mimura, H.Ninomiya, Spatial segregation limit of a competation-diffusion system with Dirichlet boundary condition, Nonlinear Anal. Real World Appl. 5(4)(2004) 645-665.

[10] E.C.M.Crooks, D.Hilhorst, Self-similar fast-reaction limits for reaction-diffusion systems on unbounded domains, J. Differential Equations, 261(3)(2016) 2210-2250.

[11] E.DiBenedetto, Continuity of weak solutions to a general porous medium equation, Indiana Univ. Math. J. 32(1)(1983) 83-118.

[12] N.Dunford, J.T.Schwartz, Linear Operators I. General Theory, Vol.7 Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London, 1958.

[13] L.C.Evans, R.F.Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, FL, 1992.

[14] D.Gilbarg, N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition, Springer-Verlag, Berlin, 2001.

[15] S.P.Hastings, J.B.McLeod, Classical Methods in Ordinary Differential Equtions, American Methematical Society, Providence, RI, 2012.

[16] D.Hilhorst, S.Martin, M.Mimura, Singular limit of a competition-diffusion system with large interspecific interaction, J. Math. Anal. Appl. 390(2)(2012) 488-513.

[17] D.Hilhorst, R.van der Hout, L.A.Peletier, The fast reaction limit for a reaction-diffusion system, J. Math. Anal. Appl. 199(2)(1996) 349-373.

[18] D.Hilhorst, R.van de Hout, L.A.Peletier, Nonlinear diffusion in the presence of fast reaction, Nonliear Anal. 41(5-6)(2000) 803-823.

[19] D.Hilhorst, J.R.King, M.Röger, Mathematical analysis of a model describing the invasion of bacteria in burn wounds, Nonliear Anal. 66(5)(2007) 1118-1140.

[20] P.Hartman, Ordinary Differential Equations, John Wiley&Sons, Inc., New York-London-Sydney, 1964.

[21] O.A.Ladyzhenskaya, V.A.Solonnikov, N.N.Uraltseva, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, Vol.23 American Mathematical Society, Providence, R.I. 1968.

[22] N.Meyers, J.Serrin, $H = W$, Proc. Nat. Acad. Sci. USA 51(1964) 1055-1056.

[23] M.H.A.Newman, Elements of the Topology of Plane Sets of Points. Cambridge, at the university press, 1951.

[24] C.J.Van Duijn, S.M.Gomes, Zhang Hongfei, On a class of similarity solution of the equation $u_t = (|u|^{m-1}u_x)_x$ with $m > -1$, IMA J. Appl. Math. 41(2)(1988) 147-163.

[25] J.L.Vazquez, The Porous Medium Equation, Mathematical Theory, The Clarendon Press, Oxford University Press, Oxford, 2007.